Cut here and keep for reference

CHAPTER 1 Review: Concept Check Answers

1. (a) What does the set of natural numbers consist of? What does the set of integers consist of? Give an example of an integer that is not a natural number.

The set of natural numbers consists of the counting numbers $1, 2, 3, \ldots$. The set of integers consists of the natural numbers together with their negatives and 0. The number -1 is an integer that is not a natural number.

(b) What does the set of rational numbers consist of? Give an example of a rational number that is not an integer.

The set of rational numbers is constructed by taking all ratios of nonzero integers, and then adding the number 0. The number $2/3$ is a rational number that is not an integer.

(c) What does the set of irrational numbers consist of? Give an example of an irrational number.

The set of irrational numbers consists of all those numbers that cannot be expressed as a ratio of integers. The number $\sqrt{5}$ is an irrational number.

(d) What does the set of real numbers consist of?

The set of real numbers consists of all the rational numbers along with all the irrational numbers.

2. A property of real numbers is given. State the property and give an example in which the property is used.

(i) Commutative Property:

$a + b = b + a$ and $ab = ba$. For example, $5 + 8 = 8 + 5$ and $5 \cdot 8 = 8 \cdot 5$.

(ii) Associative Property:

$(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$. For example, $(2 + 5) + 3 = 2 + (5 + 3)$ and $(2 \cdot 5)3 = 2(5 \cdot 3)$.

(iii) Distributive Property:

$a(b + c) = ab + ac$ and $(b + c)a = ab + ac$. For example, $7(1 + 4) = 7 \cdot 1 + 7 \cdot 4$ and $(2 + 5)9 = 9 \cdot 2 + 9 \cdot 5$.

3. Explain the difference between the open interval (a, b) and the closed interval $[a, b]$. Give an example of an interval that is neither open nor closed.

The open interval excludes the endpoints a and b, and the closed interval includes the endpoints a and b. The interval $(0, 1]$ is neither open nor closed.

4. Give the formula for finding the distance between two real numbers a and b. Use the formula to find the distance between 103 and -52.

The distance between a and b is $|b - a|$. The distance between 103 and -52 is $|(-52) - 103| = 155$.

5. Suppose $a \neq 0$ is any real number.

(a) In the expression a^n, which is the base and which is the exponent?

The base is a and the exponent is n.

(b) What does a^n mean if n is a positive integer? What does 6^5 mean?

The expression a^n means to multiply a by itself n times. For example, $6^5 = 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6$.

(c) What does a^{-n} mean if n is a positive integer? What does 3^{-2} mean?

The expression a^{-n} means the reciprocal of a^n, that is, $a^{-n} = \dfrac{1}{a^n}$. For example, $3^{-2} = \dfrac{1}{3^2}$.

(d) What does a^n mean if n is zero?

Any number raised to the 0 power is always equal to 1.

(e) If m and n are positive integers, what does $a^{m/n}$ mean? What does $4^{3/2}$ mean?

The expression $a^{m/n}$ means the nth root of the mth power of a. So $4^{3/2}$ means that you take the square root of 4 and then raise it to the third power: $4^{3/2} = 8$.

6. State the first five Laws of Exponents. Give examples in which you would use each law.

Law 1: $a^m a^n = a^{m+n}$; $\quad 5^2 \cdot 5^6 = 5^8$

Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$; $\quad \dfrac{3^4}{3^2} = 3^{4-2} = 3^2$

Law 3: $(a^m)^n = a^{mn}$; $\quad (3^2)^4 = 3^{2 \cdot 4} = 3^8$

Law 4: $(ab)^n = a^n b^n$; $\quad (3 \cdot 5)^4 = 3^4 \cdot 5^4$

Law 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$; $\quad \left(\dfrac{3}{5}\right)^2 = \dfrac{3^2}{5^2}$

7. When you multiply two powers of the same number, what should you do with the exponents? When you raise a power to a new power, what should you do with the exponents?

When you multiply two powers of the same number, you add the exponents. When you raise a power to a new power, you multiply the two exponents.

8. (a) What does $\sqrt[n]{a} = b$ mean?

The number b is the nth root of a.

(b) Is it true that $\sqrt{a^2}$ is equal to $|a|$? Try values for a that are positive and negative.

Yes, $\sqrt{a^2} = |a|$.

(c) How many real nth roots does a positive real number have if n is even? If n is odd?

There are two real nth roots if n is even and one real nth root if n is odd.

(d) Is $\sqrt[4]{-2}$ a real number? Is $\sqrt[3]{-2}$ a real number? Explain why or why not.

The expression $\sqrt[4]{-2}$ does not represent a real number because the fourth root of a negative number is undefined. The expression $\sqrt[3]{-2}$ does represent a real number because the third root of a negative number is defined.

(*continued*)

9. Explain the steps involved in rationalizing a denominator. What is the logical first step in rationalizing the denominator of the expression $\frac{5}{\sqrt{3}}$?

The logical first step in rationalizing $\frac{5}{\sqrt{3}}$ is to multiply the numerator and denominator by $\sqrt{3}$:

$$\frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

10. Explain the difference between expanding an expression and factoring an expression.

We use the Distributive Property to expand algebraic expressions, and we reverse this process by factoring an expression as a product of simpler ones.

11. State the Special Product Formulas used for expanding the given expression. Use the appropriate formula to expand $(x + 5)^2$ and $(x + 5)(x - 5)$.

(i) $(a + b)^2 = a^2 + 2ab + b^2$

(ii) $(a - b)^2 = a^2 - 2ab + b^2$

(iii) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(iv) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(v) $(a + b)(a - b) = a^2 - b^2$

By (i) we have $(x + 5)^2 = x^2 + 10x + 25$, and by (v) we have $(x + 5)(x - 5) = x^2 - 25$.

12. State the following Special Factoring Formulas. Use the appropriate formula to factor $x^2 - 9$.

(i) Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$

(ii) Perfect Square: $a^2 + 2ab + b^2 = (a + b)^2$

(iii) Sum of Cubes: $(a + b)(a^2 - ab + b^2) = (a + b)^3$

By (i) we have $x^2 - 9 = (x + 3)(x - 3)$.

13. If the numerator and the denominator of a rational expression have a common factor, how would you simplify the expression? Simplify the expression $\frac{x^2 + x}{x + 1}$.

You would simplify the expression by canceling the common factors in the numerator and the denominator. We simplify the expression as follows:

$$\frac{x^2 + x}{x + 1} = \frac{x\cancel{(x + 1)}}{\cancel{x + 1}} = x$$

14. Explain the following.

(a) How to multiply and divide rational expressions.

To multiply two rational expressions, we multiply their numerators and multiply their denominators. To divide a rational expression by another rational expression, we invert the divisor and multiply.

(b) How to add and subtract rational expressions.

To add or subtract two rational expressions, we first find the least common denominator (LCD), then rewrite the expressions using the LCD, and then add the fractions and combine the terms in the numerator.

(c) What LCD do we use to perform the addition in the expression $\frac{3}{x - 1} + \frac{5}{x + 2}$?

We use $(x - 1)(x + 2)$.

15. What is the logical first step in rationalizing the denominator of $\frac{3}{1 + \sqrt{x}}$?

Multiply both the numerator and the denominator by $(1 - \sqrt{x})$:

$$\frac{3}{1 + \sqrt{x}} \cdot \frac{1 - \sqrt{x}}{1 - \sqrt{x}} = \frac{3(1 - \sqrt{x})}{1 - x}$$

16. What is the difference between an algebraic expression and an equation? Give examples.

An algebraic expression is a combination of variables; for example, $2x^2 + xy + 6$. An equation is a statement that two mathematical expressions are equal; for example, $3x - 2y = 9x - 1$.

17. Write the general form of each type of equation.

(i) Linear equation: $ax + b = 0$

(ii) Quadratic equation: $ax^2 + bx + c = 0$

18. What are the three ways to solve a quadratic equation?

(i) Factor the equation and use the Zero-Product property.
(ii) Complete the square and solve.
(iii) Use the Quadratic Formula.

19. State the Zero-Product Property. Use the property to solve the equation $x(x - 1) = 0$.

The Zero-Product Property states that $AB = 0$ if and only if $A = 0$ or $B = 0$.

To solve the equation $x(x - 1) = 0$, the Zero-Product Property shows that either $x = 0$ or $x = 1$.

20. What do you need to add to $ax^2 + bx$ to complete the square? Complete the square for the expression $x^2 + 6x$.

To complete the square, add $\left(\frac{b}{2}\right)^2$. To make $x^2 + 6x$ a perfect square, add $\left(\frac{6}{2}\right)^2 = 9$, and this gives the perfect square $x^2 + 6x + 9 = (x + 3)^2$.

21. State the Quadratic Formula for the quadratic equation $ax^2 + bx + c = 0$, and use it to solve the equation $x^2 + 6x - 1 = 0$.

The Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Using the Quadratic Formula we get

$$x = \frac{-6 \pm \sqrt{36 - 4(1)(-1)}}{2(1)} = -3 \pm \sqrt{10}$$

(*continued*)

Cut here and keep for reference

22. What is the discriminant of the quadratic equation $ax^2 + bx + c = 0$? Find the discriminant of $2x^2 - 3x + 5 = 0$. How many real solutions does this equation have?

The discriminant is $b^2 - 4ac$. The discriminant of $2x^2 - 3x + 5 = 0$ is negative, so there are no real solutions.

23. What is the logical first step in solving the equation $\sqrt{x - 1} = x - 3$? Why is it important to check your answers when solving equations of this type?

The logical first step in solving this equation is to square both sides. It is important to check your answers because the operation of squaring both sides can turn a false equation into a true one. In this case $x = 5$ and $x = 2$ are potential solutions, but after checking, we see that $x = 5$ is the only solution.

24. What is a complex number? Give an example of a complex number, and identify the real and imaginary parts.

A complex number is an expression of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. The complex number $2 + 3i$ has real part 2 and imaginary part 3.

25. What is the complex conjugate of a complex number $a + bi$?

The complex conjugate of $a + bi$ is $a - bi$.

26. (a) How do you add complex numbers?

To add complex numbers, add the real parts and the imaginary parts.

(b) How do you multiply $(3 + 5i)(2 - i)$?

Multiply complex numbers like binomials:
$(3 + 5i)(2 - i) = 6 + 10i - 3i - 5i^2 = 11 + 7i$

(c) Is $(3 - i)(3 + i)$ a real number?

Yes, $(3 - i)(3 + i) = 9 - i^2 = 10$

(d) How do you simplify the quotient $(3 + 5i)/(3 - i)$?

Multiply the numerator and the denominator by $3 + i$, the complex conjugate of the denominator.

27. State the guidelines for modeling with equations.

(i) Identify the variable.
(ii) Translate from words to algebra.
(iii) Set up the model.
(iv) Solve the equation and check your answer.

28. Explain how to solve the given type of problem.

(a) Linear inequality: $2x \geq 1$

Divide both sides by 2; the solution set is $\left[\frac{1}{2}, \infty\right)$.

(b) Nonlinear inequality: $(x - 1)(x - 4) < 0$

Find the intervals and make a table or diagram; the solution set is $(1, 4)$.

(c) Absolute value equation: $|2x - 5| = 7$

Solve the two equations $2x - 5 = 7$ and $2x - 5 = -7$; the solutions are $x = 6$ and $x = -1$.

(d) Absolute value inequality: $|2x - 5| \leq 7$

Solve the equivalent inequality $-7 \leq 2x - 5 \leq 7$; the solution set is $[-1, 6]$.

29. (a) In the coordinate plane, what is the horizontal axis called and what is the vertical axis called?

The horizontal axis is called the x-axis and the vertical axis is called the y-axis.

(b) To graph an ordered pair of numbers (x, y), you need the coordinate plane. For the point $(2, 3)$, which is the x-coordinate and which is the y-coordinate?

The x-coordinate is 2, and the y-coordinate is 3.

(c) For an equation in the variables x and y, how do you determine whether a given point is on the graph? Is the point $(5, 3)$ on the graph of the equation $y = 2x - 1$?

Any point (x, y) on the graph must satisfy the equation. Since $3 \neq 2(5) - 1$, the point $(5, 3)$ is *not* on the graph of the equation $y = 2x - 1$.

30. (a) What is the formula for finding the distance between the points (x_1, y_1) and (x_2, y_2)?

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(b) What is the formula for finding the midpoint between (x_1, y_1) and (x_2, y_2)?

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

31. How do you find x-intercepts and y-intercepts of a graph of an equation?

To find the x-intercepts, you set $y = 0$ and solve for x. To find the y-intercepts, you set $x = 0$ and solve for y.

32. (a) Write an equation of the circle with center (h, k) and radius r.

$$(x - h)^2 + (y - k)^2 = r^2$$

(b) Find the equation of the circle with center $(2, -1)$ and radius 3.

$$(x - 2)^2 + (y + 1)^2 = 9$$

33. (a) How do you test whether the graph of an equation is symmetric with respect to the (i) x-axis, (ii) y-axis, and (iii) origin?

(i) When you replace y by $-y$, the resulting equation is equivalent to the original one.
(ii) When you replace x by $-x$, the resulting equation is equivalent to the original one.
(iii) When you replace x by $-x$ and y by $-y$, the resulting equation is equivalent to the original one.

(b) What type of symmetry does the graph of the equation $xy^2 + y^2x^2 = 3x$ have?

The graph is symmetric with respect to the x-axis.

(*continued*)

34. **(a)** What is the slope of a line? How do you compute the slope of the line through the points $(-1, 4)$ and $(1, -2)$?

The slope of a line is a measure of "steepness." The slope of the line through the points $(-1, 4)$ and $(1, -2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{-2 - 4}{1 - (-1)} = -3$$

(b) How do you find the slope and y-intercept of the line $6x + 3y = 12$?

You write the equation in slope-intercept form $y = mx + b$. The slope is m, and the y-intercept is b. The slope-intercept form of this line is $y = -2x + 4$, so the slope is -2 and the intercept is 4.

(c) How do you write the equation for a line that has slope 3 and passes through the point $(1, 2)$?

Use the point-slope form of the equation of a line. So the equation is $y - 2 = 3(x - 1)$.

35. Give an equation of a vertical line and of a horizontal line that passes through the point $(2, 3)$.

An equation of a vertical line that passes through $(2, 3)$ is $x = 2$. An equation of a horizontal line that passes through (2, 3) is $y = 3$.

36. State the general equation of a line.

$Ax + By = C$, where A and B are not both zero

37. Given lines with slopes m_1 and m_2, explain how you can tell whether the lines are (i) parallel, (ii) perpendicular.

(i) The lines are parallel if $m_1 = m_2$.

(ii) The lines are perpendicular if $m_2 = -\dfrac{1}{m_1}$.

38. How do you solve an equation (i) algebraically? (ii) graphically?

(i) Use the rules of algebra to isolate the unknown on one side of the equation.

(ii) Move all terms to one side and set that side equal to y. Sketch a graph of the resulting equation to find the values of x at which $y = 0$.

39. How do you solve an inequality (i) algebraically? (ii) graphically?

(i) Use the rules of algebra to isolate the unknown on one side of the inequality.

(ii) Move all terms to one side, and set that side equal to y. Sketch a graph to find the values x where the graph is above (or below) the x-axis.

40. Write an equation that expresses each relationship.

(a) y is directly proportional to x: $y = kx$

(b) y is inversely proportional to x: $y = \dfrac{k}{x}$

(c) z is jointly proportional to x and y: $z = kxy$

Cut here and keep for reference

CHAPTER 2 Review: Concept Check Answers

1. Define each concept.

(a) Function

A function f is a rule that assigns to each input x in a set A exactly one output $f(x)$ in a set B.

(b) Domain and range of a function

The domain of a function is the set of all the possible input values, and the range is the set of all possible output values.

(c) Graph of a function

The graph of a function f is the set of all ordered pairs $(x, f(x))$ plotted in a coordinate plane for x in the domain of f.

(d) Independent and dependent variables

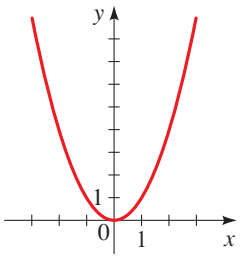
The symbol that represents any value in the domain of a function f is called an independent variable, and the symbol that represents any value in the range of f is called a dependent variable.

2. Describe the four ways of representing a function.

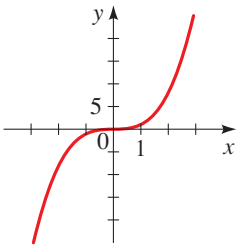
A function can be represented verbally (using words), algebraically (using a formula), visually (using a graph), and numerically (using a table of values).

3. Sketch graphs of the following functions by hand.

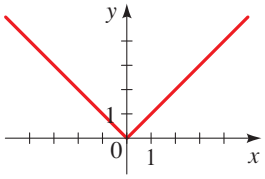
(a) $f(x) = x^2$



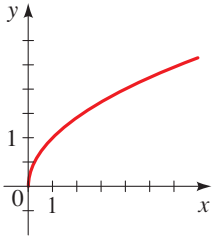
(b) $g(x) = x^3$



(c) $h(x) = |x|$



(d) $k(x) = \sqrt{x}$



4. What is a piecewise defined function? Give an example.

A piecewise defined function is defined by different formulas on different parts of its domain. An example is

$$f(x) = \begin{cases} x^2 & \text{if } x > 0 \\ 2 & \text{if } x \le 0 \end{cases}$$

5. **(a)** What is the Vertical Line Test, and what is it used for?

The Vertical Line Test states that a curve in the coordinate plane represents a function if and only if no vertical line intersects the curve more than once. It is used to determine when a given curve represents a function.

(b) What is the Horizontal Line Test, and what is it used for?

The Horizontal Line Test states that a function is one-to-one if and only if no horizontal line intersects its graph more than once. It is used to determine when a function is one-to-one.

6. Define each concept, and give an example of each.

(a) Increasing function

A function is increasing when its graph rises. More precisely, a function is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I. For example, the function $f(x) = x^2$ is an increasing function on the interval $(0, \infty)$.

(b) Decreasing function

A function is decreasing when its graph falls. More precisely, a function is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I. For example, the function $f(x) = x^2$ is a decreasing function on the interval $(-\infty, 0)$.

(c) Constant function

A function f is constant if $f(x) = c$. For example, the function $f(x) = 3$ is constant.

7. Suppose we know that the point $(3, 5)$ is a point on the graph of a function f. Explain how to find $f(3)$ and $f^{-1}(5)$.

Since $(3, 5)$ is on the graph of f, the value 3 is the input and the value 5 is the output, so $f(3) = 5$ and $f^{-1}(5) = 3$.

8. What does it mean to say that $f(4)$ is a local maximum value of f?

The value $f(4)$ is a local maximum if $f(4) \ge f(x)$ for all x near 4.

9. Explain how to find the average rate of change of a function f between $x = a$ and $x = b$.

The average rate of change of f is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

10. **(a)** What is the slope of a linear function? How do you find it? What is the rate of change of a linear function?

The slope of the graph of a linear function $f(x) = ax + b$ is the same as the rate of change of f, and they are both equal to a, the coefficient of x.

(b) Is the rate of change of a linear function constant? Explain.

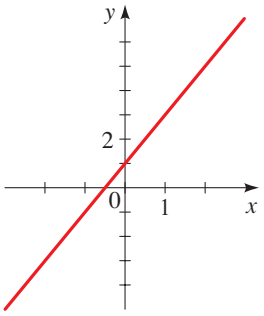
Yes, because it is equal to the slope, and the slope is the same between any two points.

(*continued*)

(c) Give an example of a linear function, and sketch its graph.

An example is $f(x) = 2x + 1$, and the graph is shown below.

$$f(x) = 2x + 1$$



11. Suppose the graph of a function f is given. Write an equation for each of the graphs that are obtained from the graph of f as follows.

(a) Shift upward 3 units: $y = f(x) + 3$

(b) Shift downward 3 units: $y = f(x) - 3$

(c) Shift 3 units to the right: $y = f(x - 3)$

(d) Shift 3 units to the left: $y = f(x + 3)$

(e) Reflect in the x-axis: $y = -f(x)$

(f) Reflect in the y-axis: $y = f(-x)$

(g) Stretch vertically by a factor of 3: $y = 3f(x)$

(h) Shrink vertically by a factor of $\frac{1}{3}$: $y = \frac{1}{3}f(x)$

(i) Shrink horizontally by a factor of $\frac{1}{3}$: $y = f(3x)$

(j) Stretch horizontally by a factor of 3: $y = f(\frac{1}{3}x)$

12. (a) What is an even function? How can you tell that a function is even by looking at its graph? Give an example of an even function.

An even function f satisfies $f(-x) = f(x)$ for all x in its domain. If the graph of a function is symmetric with respect to the y-axis, then the function is even. Some examples are $f(x) = x^2$ and $f(x) = |x|$.

(b) What is an odd function? How can you tell that a function is odd by looking at its graph? Give an example of an odd function.

An odd function f satisfies $f(-x) = -f(x)$ for all x in its domain. If the graph of a function is symmetric with respect to the origin, then the function is odd. Some examples are $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$.

13. Suppose that f has domain A and g has domain B. What are the domains of the following functions?

(a) Domain of $f + g$: $A \cap B$

(b) Domain of fg: $A \cap B$

(c) Domain of f/g: $\{x \in A \cap B \mid g(x) \neq 0\}$

14. (a) How is the composition function $f \circ g$ defined? What is its domain?

The function $f \circ g$ is defined by $f \circ g(x) = f(g(x))$. The domain is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

(b) If $g(a) = b$ and $f(b) = c$, then explain how to find $(f \circ g)(a)$.

To find $f \circ g(a)$, we evaluate the following:

$$f \circ g(a) = f(g(a)) = f(b) = c$$

15. (a) What is a one-to-one function?

A function with domain A is called a one-to-one function if no two elements of A have the same image. More precisely, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

(b) How can you tell from the graph of a function whether it is one-to-one?

We use the Horizontal Line Test, which states that a function is one-to-one if and only if no horizontal line intersects its graph more than once.

(c) Suppose that f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What are the domain and range of f^{-1}?

The inverse function of f has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

(d) If you are given a formula for f, how do you find a formula for f^{-1}? Find the inverse of the function $f(x) = 2x$.

We write $y = f(x)$, solve the equation for x in terms of y, and interchange x and y. The resulting equation is $y = f^{-1}(x)$. If $f(x) = 2x$, we write $y = 2x$, solve for x to get $x = \frac{1}{2}y$, interchange x and y to get $f^{-1}(x) = \frac{1}{2}x$.

(e) If you are given a graph of f, how do you find a graph of the inverse function f^{-1}?

The graph of the inverse function f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Cut here and keep for reference

CHAPTER 3 Review: Concept Check Answers

1. (a) What is the degree of a quadratic function f? What is the standard form of a quadratic function? How do you put a quadratic function into standard form?

A quadratic function f is a polynomial of degree 2. The standard form of a quadratic function f is $f(x) = a(x - h)^2 + k$. Complete the square to put a quadratic function into standard form.

(b) The quadratic function $f(x) = a(x - h)^2 + k$ is in standard form. The graph of f is a parabola. What is the vertex of the graph of f? How do you determine whether $f(h) = k$ is a minimum or a maximum value?

The vertex of the graph of f is (h, k). If the coefficient a is positive, then the graph of f opens upward and $f(h) = k$ is a minimum value. If a is negative, then the graph of f opens downward and $f(h) = k$ is a maximum value.

(c) Express $f(x) = x^2 + 4x + 1$ in standard form. Find the vertex of the graph and the maximum or minimum value of f.

We complete the square to get $f(x) = (x + 2)^2 - 3$. The graph is a parabola that opens upward with vertex $(-2, -3)$. The minimum value is $f(-2) = -3$.

2. (a) Give the general form of polynomial function P of degree n.

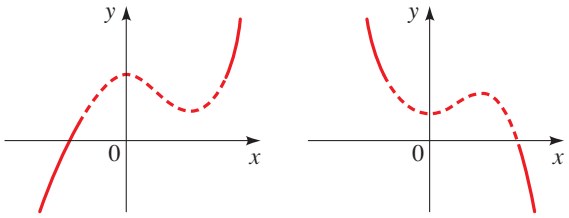
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

(b) What does it mean to say that c is a zero of P? Give two equivalent conditions that tell us that c is a zero of P.

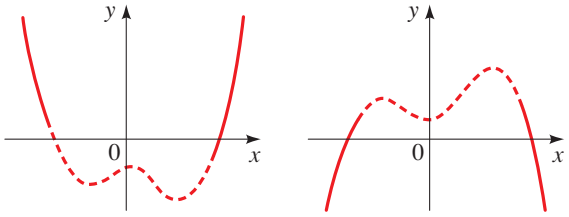
The value c is a zero of P if $P(c) = 0$. Equivalently, c is a zero of P if $x - c$ is a factor of P or if c is an x-intercept of the graph of P.

3. Sketch graphs showing the possible end behaviors of polynomials of odd degree and of even degree.

Odd degree



Even degree



4. What steps do you follow to graph a polynomial function P?

We first find the zeros of P and then make a table using test points between successive zeros. We then determine the end behavior and use all this information to graph P.

5. (a) What is a local maximum point or local minimum point of a polynomial P?

The point $(a, P(a))$ is a local maximum if it is the highest point on the graph of P within some viewing rectangle. The point $(b, P(b))$ is a local minimum if it is the lowest point on the graph of P within some viewing rectangle.

(b) How many local extrema can a polynomial P of degree n have?

The graph of P has at most $n - 1$ local extrema.

6. When we divide a polynomial $P(x)$ by a divisor $D(x)$, the Division Algorithm tells us that we can always obtain a quotient $Q(x)$ and a remainder $R(x)$. State the two forms in which the result of this division can be written.

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$P(x) = D(x)Q(x) + R(x)$$

7. (a) State the Remainder Theorem.

If a polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

(b) State the Factor Theorem.

The number c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

(c) State the Rational Zeros Theorem.

If the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then every rational zero of P is of the form p/q, where p is a factor of the constant coefficient a_0 and q is a factor of the leading coefficient a_n.

8. What steps would you take to find the rational zeros of a polynomial P?

First list all possible rational zeros of P given by the Rational Zeros Theorem. Evaluate P at a possible zero (using synthetic division), and note the quotient if the remainder is 0. Repeat this process on the quotient until you reach a quotient that is quadratic. Then use the quadratic formula to find the remaining zeros.

(*continued*)

9. Let $P(x) = 2x^4 - 3x^3 + x - 15$.

(a) Explain how Descartes' Rule of Signs is used to determine the possible number of positive and negative real roots of P.

Since there are three variations in sign in $P(x)$, by Descartes' Rule of Signs there are either three or one positive real zeros. Since there is one variation in sign in $P(-x)$, by Descartes' Rule of Signs there is exactly one negative real zero.

(b) What does it mean to say that a is a lower bound and b is an upper bound for the zeros of a polynomial?

We say that a is a lower bound and b is an upper bound for the zeros of a polynomial if every real zero c of the polynomial satisfies $a \le c \le b$.

(c) Explain how the Upper and Lower Bounds Theorem is used to show that all the real zeros of P lie between -3 and 3.

When we divide P by $x - 3$, the row that contains the quotient and the remainder has only nonnegative entries, so 3 is an upper bound. When we divide P by $x - (-3) = x + 3$, the row that contains the quotient and the remainder has entries that alternate in sign, so -3 is a lower bound.

10. (a) State the Fundamental Theorem of Algebra.

Every polynomial has at least one complex zero.

(b) State the Complete Factorization Theorem.

Every polynomial of degree $n \ge 1$ can be factored completely into linear factors (with complex coefficients).

(c) State the Zeros Theorem.

Every polynomial of degree $n \ge 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

(d) State the Conjugate Zeros Theorem.

If a polynomial has real coefficients and if the complex number z is a zero of the polynomial, then its complex conjugate $\bar{z}$ is also a zero of the polynomial.

11. (a) What is a rational function?

A rational function is a function of the form $r(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.

(b) What does it mean to say that $x = a$ is a vertical asymptote of $y = f(x)$?

The line $x = a$ is a vertical asymptote if

$$y \to \pm\infty \quad \text{as} \quad x \to a^+ \quad \text{or} \quad x \to a^-$$

(c) What does it mean to say that $y = b$ is a horizontal asymptote of $y = f(x)$?

The line $y = b$ is a horizontal asymptote if

$$y \to b \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

(d) Find the vertical and horizontal asymptotes of

$$f(x) = \frac{5x^2 + 3}{x^2 - 4}$$

The denominator factors as $(x - 2)(x + 2)$, so the vertical asymptotes are $x = 2$ and $x = -2$. The horizontal asymptote is $y = 5$.

12. (a) How do you find vertical asymptotes of rational functions?

Vertical asymptotes of a rational function are the line $x = a$, where a is a zero of the denominator.

(b) Let s be the rational function

$$s(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

How do you find the horizontal asymptote of s?

If $n < m$, then the horizontal asymptote is

$$y = 0$$

If $n = m$, then the horizontal asymptote is

$$y = \frac{a_n}{b_m}$$

If $n > m$, then there is no horizontal asymptote.

13. (a) Under what circumstances does a rational function have a slant asymptote?

If $r(x) = P(x)/Q(x)$ and the degree of P is one greater than the degree of Q, then r has a slant asymptote.

(b) How do you determine the end behavior of a rational function?

Divide the numerator by the denominator; the quotient determines the end behavior of the function.

14. (a) Explain how to solve a polynomial inequality.

Move all terms to one side, factor the polynomial, find the zeros of the polynomial, use the zeros and test points to make a sign diagram, and use the diagram to solve the inequality.

(b) What are the cut points of a rational function? Explain how to solve a rational inequality.

The cut points are the zeros of the numerator and zeros of the denominator. To solve a rational inequality, move all terms to one side, factor the numerator and denominator to find all the cut points, use the cut points and test points to make a sign diagram, and use the diagram to solve the inequality.

(c) Solve the inequality $x^2 - 9 \le 8x$.

Move all terms to one side and then factor: $(x + 1)(x - 9) \le 0$. We make a sign diagram as shown.

	-1		9	
Sign of $x + 1$	$-$	$+$		$+$
Sign of $x - 9$	$-$		$-$	$+$
Sign of $(x + 1)(x - 9)$	$+$	$-$		$+$

The solution is the interval $[-1, 9]$.

Cut here and keep for reference

CHAPTER 4 Review: Concept Check Answers

1. Let f be the exponential function with base a.

(a) Write an equation that defines f.

$$f(x) = a^x$$

(b) Write an equation for the exponential function f with base 3.

$$f(x) = 3^x$$

2. Let f be the exponential function $f(x) = a^x$, where $a > 0$.

(a) What is the domain of f?

All real numbers $(-\infty, \infty)$

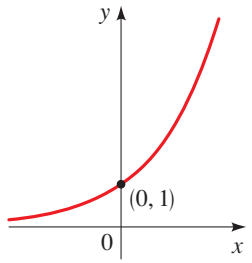
(b) What is the range of f?

All positive real numbers $(0, \infty)$

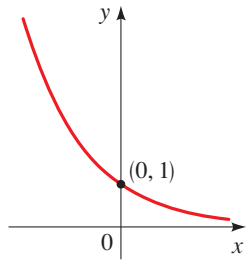
(c) Sketch graphs of f for the following cases.

(i) $a > 1$ (ii) $0 < a < 1$

(i)



(ii)



3. If x is large, which function grows faster, $f(x) = 2^x$ or $g(x) = x^2$?

The function $f(x) = 2^x$ grows faster. We can see this by graphing both functions in a sufficiently large viewing rectangle.

4. **(a)** How is the number e defined?

The number e is the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as n becomes large.

(b) Give an approximate value of e, correct to four decimal places.

$$e \approx 2.71828$$

(c) What is the natural exponential function?

It is the exponential function with base e:

$$f(x) = e^x$$

5. **(a)** How is $\log_a x$ defined?

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

(b) Find $\log_3 9$.

$$\log_3 9 = 2 \quad \text{because} \quad 3^2 = 9$$

(c) What is the natural logarithm?

It is the logarithm with base e: $\ln x = \log_e x$

(d) What is the common logarithm?

It is the logarithm with base 10: $\log x = \log_{10} x$

(e) Write the exponential form of the equation $\log_7 49 = 2$.

$$7^2 = 49$$

6. Let f be the logarithmic function $f(x) = \log_a x$.

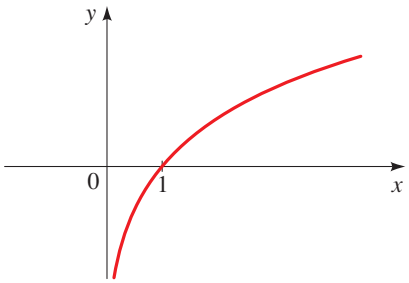
(a) What is the domain of f?

All positive real numbers $(0, \infty)$

(b) What is the range of f?

All real numbers $(-\infty, \infty)$

(c) Sketch a graph of the logarithmic function for the case that $a > 1$.



7. State the three Laws of Logarithms.

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^N = N \log_a x$$

8. **(a)** State the Change of Base Formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

(b) Find $\log_7 30$.

By the Change of Base Formula

$$\log_7 30 = \frac{\log 30}{\log 7} \approx 1.7479$$

9. **(a)** What is an exponential equation?

An exponential equation is one in which the unknown occurs in an exponent.

(b) How do you solve an exponential equation?

First isolate the exponential term on one side, take logarithms of each side, and then use the laws of logarithms to bring down the exponent. Then solve for the unknown.

(c) Solve for x: $2^x = 19$

$$\log 2^x = \log 19$$

$$x \log 2 = \log 19$$

$$x = \frac{\log 19}{\log 2} \approx 4.2479$$

(*continued*)

10. **(a)** What is a logarithmic equation?

A logarithmic equation is one in which a logarithm of the unknown occurs.

(b) How do you solve a logarithmic equation?

First combine the logarithmic terms on one side of the equation, write the resulting equation in exponential form, and then solve for the unknown.

(c) Solve for x: $4 \log_3 x = 7$

$$4 \log_3 x = 7$$
$$\log_3 x = 1.75$$
$$x = 3^{1.75} \approx 6.84$$

11. Suppose that an amount P is invested at an interest rate r and $A(t)$ is the amount of the investment after t years. Write a formula for $A(t)$ in the following cases.

(a) Interest is compounded n times per year.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

(b) Interest is compounded continuously.

$$A(t) = Pe^{rt}$$

12. Suppose that the initial size of a population is n_0 and the population grows exponentially. Let $n(t)$ be the size of the population at time t.

(a) Write a formula for $n(t)$ in terms of the doubling time a.

$$n(t) = n_0 2^{t/a}$$

(b) Write a formula for $n(t)$ in terms of the relative growth rate r.

$$n(t) = n_0 e^{rt}$$

13. Suppose that the initial mass of a radioactive substance is m_0 and the half-life of the substance is h. Let $m(t)$ be the mass remaining at time t.

(a) What is meant by the half-life h?

The time it takes for a mass to decay to half its amount

(b) Write a formula for $m(t)$ in terms of the half-life h.

$$m(t) = m_0 2^{-t/h}$$

(c) Write a formula for the relative decay rate r in terms of the half-life h.

$$r = \frac{\ln 2}{h}$$

(d) Write a formula for $m(t)$ in terms of the relative decay rate r.

$$m(t) = m_0 e^{-rt}$$

14. Suppose that the initial temperature difference between an object and its surroundings is D_0 and the surroundings have temperature T_s. Let $T(t)$ be the temperature at time t. State Newton's Law of Cooling for $T(t)$.

$$T(t) = T_s + D_0 e^{-kt}$$

where k is a constant that depends on the type of object.

15. What is a logarithmic scale? If we use a logarithmic scale with base 10, what do the following numbers correspond to on the logarithmic scale?

(i) 100 (ii) 100,000 (iii) 0.0001

On a logarithmic scale, numbers are represented by their logarithms.

(i) 2 (ii) 5 (iii) -4

16. **(a)** What does the pH scale measure?

The acidity (or alkalinity) of a substance

(b) Define the pH of a substance with hydrogen ion concentration of $[H^+]$.

$$\text{pH} = -\log [H^+]$$

17. **(a)** What does the Richter scale measure?

The magnitude of earthquakes

(b) Define the magnitude M of an earthquake in terms of the intensity I of the earthquake and the intensity S of a standard earthquake.

$$M = -\log \frac{I}{S}$$

18. **(a)** What does the decibel scale measure?

The loudness of sound

(b) Define the decibel level B of a sound in terms of the intensity I of the sound and the intensity I_0 of a barely audible sound.

$$B = 10 \log \frac{I}{I_0}$$

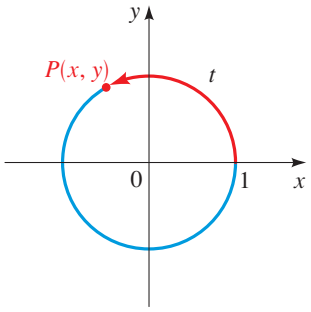
Cut here and keep for reference

CHAPTER 5 Review: Concept Check Answers

1. (a) What is the unit circle, and what is the equation of the unit circle?

The unit circle is the circle of radius 1 centered at $(0, 0)$. The equation of the unit circle is $x^2 + y^2 = 1$.

(b) Use a diagram to explain what is meant by the terminal point $P(x, y)$ determined by t.



(c) Find the terminal point for $t = \frac{\pi}{2}$.

$$P(x, y) = (0, 1)$$

(d) What is the reference number associated with t?

The reference number is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

(e) Find the reference number and terminal point for $t = \frac{7\pi}{4}$.

The reference number is $\frac{\pi}{4}$. The terminal point is in Quadrant IV, so $P(x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

2. Let t be a real number, and let $P(x, y)$ be the terminal point determined by t.

(a) Write equations that define $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, and $\cot t$.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \tfrac{y}{x}$$
$$\csc t = \tfrac{1}{y} \qquad \sec t = \tfrac{1}{x} \qquad \cot t = \tfrac{x}{y}$$

(b) In each of the four quadrants, identify the trigonometric functions that are positive.

In Quadrant I all functions are positive; in Quadrant II the sine and cosecant functions are positive; in Quadrant III the tangent and cotangent functions are positive; and in Quadrant IV the cosine and secant functions are positive.

(c) List the special values of sine, cosine, and tangent.

$\sin 0 = 0$, $\sin \frac{\pi}{6} = \frac{1}{2}$, $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin \frac{\pi}{2} = 1$

$\cos 0 = 1$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{\pi}{2} = 0$

$\tan 0 = 0$, $\tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$, $\tan \frac{\pi}{4} = 1$, $\tan \frac{\pi}{3} = \sqrt{3}$

3. (a) Describe the steps we use to find the value of a trigonometric function at a real number t.

We find the reference number for t, the quadrant where the terminal point lies, and the sign of the function in that quadrant, and we use all these to find the value of the function at t.

(b) Find $\sin \frac{5\pi}{6}$.

The terminal point of $\frac{5\pi}{6}$ is in Quadrant II. Since sine is positive in Quadrant II, $\sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2}$.

4. (a) What is a periodic function?

A function f is periodic if there is a positive number p such that $f(x + p) = f(x)$ for every x. The least such p is called the period of f.

(b) What are the periods of the six trigonometric functions?

The sine, cosine, cosecant, and secant functions have period 2π, and the tangent and cotangent functions have period π.

(c) Find $\sin \frac{19\pi}{4}$.

$$\sin \tfrac{19\pi}{4} = \sin\left(\tfrac{3\pi}{4} + 4\pi\right) = \sin \tfrac{3\pi}{4} = \tfrac{\sqrt{2}}{2}$$

5. (a) What is an even function, and what is an odd function?

An even function satisfies $f(-x) = f(x)$.

An odd function satisfies $f(-x) = -f(x)$.

(b) Which trigonometric functions are even? Which are odd?

The cosine and secant functions are even; the sine, cosecant, tangent, and cotangent functions are odd.

(c) If $\sin t = 0.4$, find $\sin(-t)$.

Since the sine function is odd, $\sin(-t) = -0.4$.

(d) If $\cos s = 0.7$, find $\cos(-s)$.

Since the cosine function is even, $\cos(-s) = 0.7$.

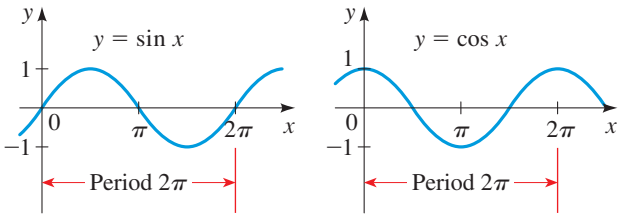
6. (a) State the reciprocal identities.

$\csc t = \frac{1}{\sin t}$, $\sec t = \frac{1}{\cos t}$, $\cot t = \frac{1}{\tan t}$,
$\tan t = \frac{\sin t}{\cos t}$, $\cot t = \frac{\cos t}{\sin t}$

(b) State the Pythagorean identities.

$\sin^2 t + \cos^2 t = 1$, $\tan^2 t + 1 = \sec^2 t$,
$1 + \cot^2 t = \csc^2 t$

7. (a) Graph the sine and cosine functions.



(b) What are the amplitude, period, and horizontal shift for the sine curve $y = a \sin k(x - b)$ and for the cosine curve $y = a \cos k(x - b)$?

Amplitude a; period $\frac{2\pi}{k}$; horizontal shift b

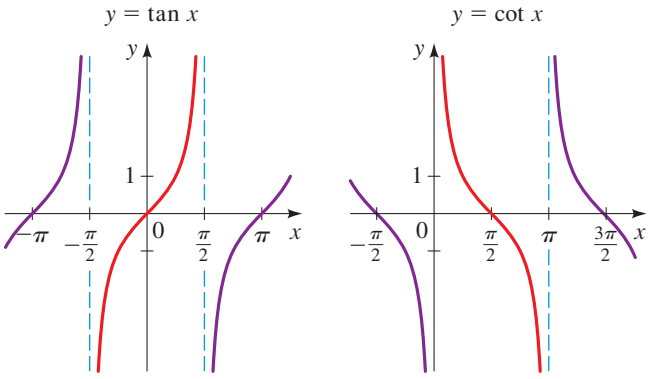
(c) Find the amplitude, period, and horizontal shift of $y = 3 \sin\left(2x - \frac{\pi}{6}\right)$.

We factor to get $y = 3 \sin 2(x - \frac{\pi}{12})$.

Amplitude 3; period π; horizontal shift $\frac{\pi}{12}$

(*continued*)

8. (a) Graph the tangent and cotangent functions.



(b) For the curves $y = a \tan kx$ and $y = a \cot kx$, state appropriate intervals to graph one complete period of each curve.

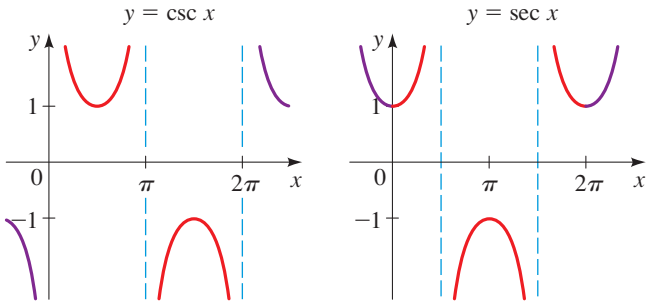
An appropriate interval for $y = a \tan kx$ is $(-\pi/2k, \pi/2k)$.

An appropriate interval for $y = a \cot kx$ is $(0, \pi/k)$.

(c) Find an appropriate interval to graph one complete period of $y = 5 \tan 3x$.

An appropriate interval for $y = 5 \tan 3x$ is $(-\pi/6, \pi/6)$.

9. (a) Graph the cosecant and secant functions.



(b) For the curves $y = a \csc kx$ and $y = a \sec kx$, state appropriate intervals to graph one complete period of each curve.

An appropriate interval for $y = a \csc kx$ is $(0, 2\pi/k)$.

An appropriate interval for $y = a \sec kx$ is $(0, 2\pi/k)$.

(c) Find an appropriate interval to graph one period of $y = 3 \csc 6x$.

An appropriate interval for $y = 3 \csc 6x$ is $(0, \pi/3)$.

10. (a) Define the inverse sine function, the inverse cosine function, and the inverse tangent function.

$$\sin^{-1}x = y \quad \Leftrightarrow \quad \sin y = x$$
$$\cos^{-1}x = y \quad \Leftrightarrow \quad \cos y = x$$
$$\tan^{-1}x = y \quad \Leftrightarrow \quad \tan y = x$$

(b) Find $\sin^{-1}\frac{1}{2}$, $\cos^{-1}\frac{\sqrt{2}}{2}$, and $\tan^{-1}1$.

From 2(c) and the definitions in part (a) we get $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, and $\tan^{-1}1 = \frac{\pi}{4}$.

(c) For what values of x is the equation $\sin(\sin^{-1}x) = x$ true? For what values of x is the equation $\sin^{-1}(\sin x) = x$ true?

$$\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}$$

11. (a) What is simple harmonic motion?

An object is in simple harmonic motion if its displacement y at time t is modeled by $y = a \sin \omega t$ or $y = a \cos \omega t$.

(b) What is damped harmonic motion?

An object is in damped harmonic motion if its displacement y at time t is modeled by $y = ke^{-ct} \sin \omega t$ or $y = ke^{-ct} \cos \omega t$, $c > 0$.

(c) Give real-world examples of harmonic motion.

The motion of a vibrating mass on a spring, the vibrations of a violin string, the brightness of a variable star, and many more

12. Suppose that an object is in simple harmonic motion given by $y = 5 \sin\left(2t - \frac{\pi}{3}\right)$.

(a) Find the amplitude, period, and frequency.

Amplitude 5; period $\frac{2\pi}{2} = \pi$; frequency $\frac{2}{2\pi} = \frac{1}{\pi}$

(b) Find the phase and the horizontal shift.

The phase is $\frac{\pi}{3}$, and the horizontal shift (or lag time) is $\frac{\pi}{6}$.

13. Consider the following models of harmonic motion.

$$y_1 = 5 \sin(2t - 1) \qquad y_2 = 5 \sin(2t - 3)$$

Do both motions have the same frequency? What is the phase for each equation? What is the phase difference? Are the objects moving in phase or out of phase?

Both motions have the same frequency: $1/\pi$. The phase of the first is 1, and the phase of the second is 3. The phase difference is $3 - 1 = 2$, which is not a multiple of 2π, so the objects are moving out of phase.

Cut here and keep for reference

CHAPTER 6 Review: Concept Check Answers

1. (a) How is the degree measure of an angle defined?

An angle of 1° is $\frac{1}{360}$ of a complete revolution.

(b) How is the radian measure of an angle defined?

The radian measure of an angle is the length of the arc that the angle subtends in a circle of radius 1.

(c) How do you convert from degrees to radians? Convert 45° to radians.

To convert from degrees to radians, we multiply by $\pi/180$. So

$$45° = 45\left(\frac{\pi}{180}\right)\text{rad} = \frac{\pi}{4}$$

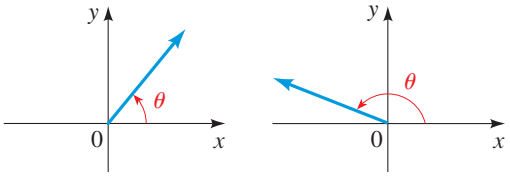
(d) How do you convert from radians to degrees? Convert 2 rad to degrees.

To convert from radians to degrees, we multiply by $180/\pi$. So

$$2 \text{ rad} = 2\left(\frac{180}{\pi}\right) \approx 114.6°$$

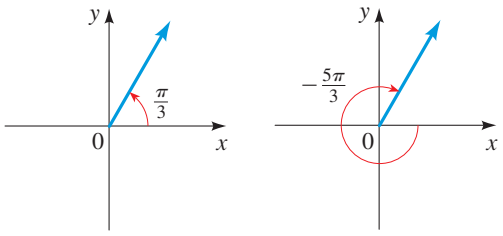
2. (a) When is an angle in standard position? Illustrate with a graph.

An angle is in standard position if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis.



(b) When are two angles in standard position coterminal? Illustrate with a graph.

Two angles are coterminal if their sides coincide. Angles that differ by a multiple of 2π rad (or a multiple of 360°) are coterminal.



(c) Are the angles 25° and 745° coterminal?

Yes, because $745° - 25° = 720°$, which is a multiple of 360°.

(d) How is the reference angle for an angle θ defined?

The reference angle $\bar{\theta}$ is the acute angle formed by the terminal side of θ and the x-axis.

(e) Find the reference angle for 150°.

The reference angle is $\bar{\theta} = 180° - 150° = 30°$.

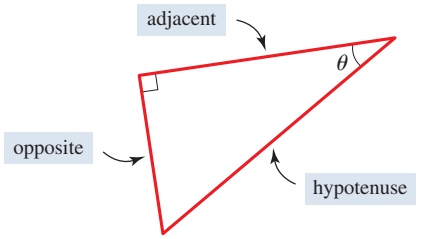
3. (a) In a circle of radius r, what is the length s of an arc that subtends a central angle of θ radians?

$$s = r\theta$$

(b) In a circle of radius r, what is the area A of a sector with central angle θ radians?

$$A = \tfrac{1}{2}r^2\theta$$

4. (a) Let θ be an acute angle in a right triangle. Identify the opposite side, the adjacent side, and the hypotenuse in the figure.

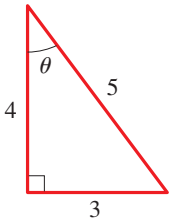


(b) Define the six trigonometric ratios in terms of the adjacent and opposite sides and the hypotenuse.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} \quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

(c) Find the six trigonometric ratios for the angle θ shown in the figure.



$$\sin\theta = \frac{3}{5} \quad \cos\theta = \frac{4}{5} \quad \tan\theta = \frac{3}{4}$$

$$\csc\theta = \frac{5}{3} \quad \sec\theta = \frac{5}{4} \quad \cot\theta = \frac{4}{3}$$

(d) List the special values of sine, cosine, and tangent.

$$\sin\tfrac{\pi}{6} = \tfrac{1}{2} \quad \sin\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2} \quad \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}$$

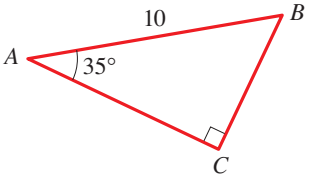
$$\cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} \quad \cos\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2} \quad \cos\tfrac{\pi}{3} = \tfrac{1}{2}$$

$$\tan\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{3} \quad \tan\tfrac{\pi}{4} = 1 \quad \tan\tfrac{\pi}{3} = \sqrt{3}$$

5. (a) What does it mean to solve a triangle?

To solve a triangle means to find all three angles and all three sides.

(b) Solve the triangle shown.



$$\angle B = 90° - 35° = 55°$$

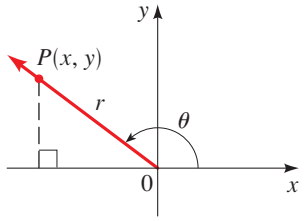
$$a = 10 \sin 35° \approx 5.74$$

$$b = 10 \cos 35° \approx 8.19$$

6. (a) Let θ be an angle in standard position, let $P(x, y)$ be a point on the terminal side, and let r be the distance from

(*continued*)

the origin to P, as shown in the figure. Write expressions for the six trigonometric functions of θ.

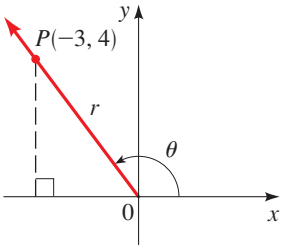


$\sin\theta = \frac{y}{r}$ $\quad \csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\quad \sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\quad \cot\theta = \frac{x}{y}$

(b) Find the sine, cosine, and tangent for the angle θ shown in the figure.



Here $x = -3$, $y = 4$, and $r = \sqrt{(-3)^2 + 4^2} = 5$. So $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5}$, and $\tan\theta = \frac{4}{-3}$.

7. In each of the four quadrants, identify the trigonometric functions that are positive.

In Quadrant I all the trigonometric functions are positive; in Quadrant II the sine and cosecant functions are positive; in Quadrant III the tangent and cotangent functions are positive; and in Quadrant IV the cosine and secant functions are positive.

8. (a) Describe the steps we use to find the value of a trigonometric function of an angle θ.

We find the reference angle for θ, the quadrant where the terminal side lies, and the sign of the function in that quadrant, and we use all these to find the value of the function at θ.

(b) Find $\sin 5\pi/6$.

The terminal side of the angle $\frac{5\pi}{6}$ is in Quadrant II, and the reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. Since sine is positive in Quadrant II, $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$.

9. (a) State the reciprocal identities.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

(b) State the Pythagorean identities.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

10. (a) What is the area of a triangle with sides of length a and b and with included angle θ?

The area is $\mathcal{A} = \frac{1}{2}ab\sin\theta$.

(b) What is the area of a triangle with sides of length a, b, and c?

The area is given by Heron's Formula

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the semiperimeter.

11. (a) Define the inverse sine function, the inverse cosine function, and the inverse tangent function.

$$\sin^{-1}x = y \iff \sin y = x$$

$$\cos^{-1}x = y \iff \cos y = x$$

$$\tan^{-1}x = y \iff \tan y = x$$

(b) Find $\sin^{-1}\frac{1}{2}$, $\cos^{-1}(\sqrt{2}/2)$, and $\tan^{-1}1$.

From 2(c) and the definitions in part (a) we get

$$\sin^{-1}\tfrac{1}{2} = \tfrac{\pi}{6} \qquad \cos^{-1}\tfrac{\sqrt{2}}{2} = \tfrac{\pi}{4} \qquad \tan^{-1}1 = \tfrac{\pi}{4}$$

(c) For what values of x is the equation $\sin(\sin^{-1}x) = x$ true? For what values of x is the equation $\sin^{-1}(\sin x) = x$ true?

$$\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1$$

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}$$

12. (a) State the Law of Sines.

In triangle ABC we have $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

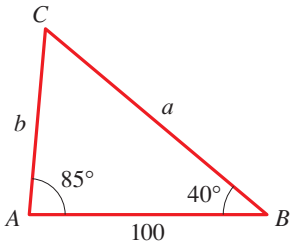
(b) Find side a in the figure.

Note that $\angle C = 180° - (85° + 40°) = 55°$.

By the Law of Sines

$$\frac{\sin 85°}{a} = \frac{\sin 55°}{100}, \text{ so}$$

$$a = \frac{100\sin 85°}{\sin 55°} \approx 121.6.$$



(c) Explain the ambiguous case in the Law of Sines.

In the case SSA there may be two triangles, one triangle, or no triangle with the given sides and angles.

13. (a) State the Law of Cosines.

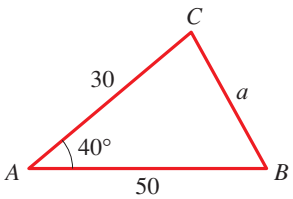
In triangle ABC we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

(b) Find side a in the figure.



By the Law of Cosines we have

$$a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$= \sqrt{50^2 + 30^2 - 2(50)(30)\cos 40°}$$

$$\approx 33.2$$

Cut here and keep for reference

CHAPTER 7 Review: Concept Check Answers

1. What is an identity? What is a trigonometric identity?

An identity is an equation that is true for all values of the variable(s). A trigonometric identity is an identity that involves trigonometric functions.

2. (a) State the Pythagorean identities.

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

(b) Use a Pythagorean identity to express cosine in terms of sine.

By the first Pythagorean identity we have
$\cos x = \pm\sqrt{1 - \sin^2 x}$

3. (a) State the reciprocal identities for cosecant, secant, and cotangent.

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

(b) State the even-odd identities for sine and cosine.

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

(c) State the cofunction identities for sine, tangent, and secant.

$$\sin\left(\tfrac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\tfrac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\tfrac{\pi}{2} - x\right) = \csc x$$

(d) Suppose that $\cos(-x) = 0.4$; use the identities in parts (a) and (b) to find $\sec x$.

$$\sec x = \frac{1}{\cos x} = \frac{1}{\cos(-x)} = \frac{1}{0.4} = 2.5$$

(e) Suppose that $\sin 10° = a$; use the identities in part (c) to find $\cos 80°$.

Since 10° and 80° are complementary angles, we have $\cos 80° = \sin 10° = a$.

4. (a) How do you prove an identity?

Start with one side of the equation, and then use known identities to transform it to the other side.

(b) Prove the identity $\sin x(\csc x - \sin x) = \cos^2 x$

$$\begin{aligned} \text{LHS} &= \sin x(\csc x - \sin x) \\ &= \sin x\left(\frac{1}{\sin x} - \sin x\right) && \text{Reciprocal identity} \\ &= 1 - \sin^2 x && \text{Distributive Property} \\ &= \cos^2 x = \text{RHS} && \text{Pythagorean identity} \end{aligned}$$

5. (a) State the Addition and Subtraction Formulas for Sine and Cosine.

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

(b) Use a formula from part (a) to find $\sin 75°$.

$$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin 45° \cos 30° + \cos 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

6. (a) State the formula for $A \sin x + B \cos x$.

Let $k = \sqrt{A^2 + B^2}$; then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where ϕ satisfies $\cos\phi = A/\sqrt{A^2 + B^2}$ and $\sin\phi = B/\sqrt{A^2 + B^2}$.

(b) Express $3 \sin x + 4 \cos x$ as a function of sine only.

We have $k = \sqrt{3^2 + 4^2} = 5$. The angle ϕ satisfies $\cos\phi = \frac{3}{5}$ and $\sin\phi = \frac{4}{5}$, so ϕ is in Quadrant I. We find $\phi = \sin^{-1}\left(\frac{4}{5}\right) \approx 53.1°$. Thus

$$3 \sin x + 4 \cos x = 5 \sin(x + 53.1°)$$

7. (a) State the Double-Angle Formula for Sine and the Double-Angle Formulas for Cosine.

$$\begin{aligned} \sin 2x &= 2 \sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2 \sin^2 x \\ &= 2 \cos^2 x - 1 \end{aligned}$$

(b) Prove the identity $\sec x \sin 2x = 2 \sin x$.

$$\begin{aligned} \text{LHS} &= \sec x \sin 2x \\ &= \sec x(2 \sin x \cos x) && \text{Double-Angle Formula} \\ &= \frac{1}{\cos x}(2 \sin x \cos x) && \text{Reciprocal identity} \\ &= 2 \sin x = \text{RHS} && \text{Pythagorean identity} \end{aligned}$$

8. (a) State the formulas for lowering powers of sine and cosine.

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

(b) Prove the identity $4 \sin^2 x \cos^2 x = \sin^2 2x$.

$$\begin{aligned} \text{LHS} &= 4 \sin^2 x \cos^2 x \\ &= 4\left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) && \text{Lower powers} \\ &= 1 - \cos^2 2x && \text{Simplify} \\ &= \sin^2 2x = \text{RHS} && \text{Pythagorean identity} \end{aligned}$$

9. (a) State the Half-Angle Formulas for Sine and Cosine.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

(b) Find $\cos 15°$.

$$\begin{aligned} \cos 15° &= \cos\left(\frac{30°}{2}\right) \\ &= \pm\sqrt{\frac{1 + \cos 30°}{2}} = \pm\sqrt{\frac{1 + \sqrt{3}/2}{2}} \\ &= \pm\sqrt{\frac{2 + \sqrt{3}}{4}} = \pm\frac{1}{2}\sqrt{2 + \sqrt{3}} \end{aligned}$$

Since 15° is in Quadrant I and since cosine is positive in Quadrant I, we conclude that $\cos 15° = \frac{1}{2}\sqrt{2 + \sqrt{3}}$.

(*continued*)

10. (a) State the Product-to-Sum Formula for the product $\sin u \cos v$.

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

(b) Express $\sin 5x \cos 3x$ as a sum of trigonometric functions.

By the formula in part (a) we have

$$\begin{aligned}\sin 5x \cos 3x &= \tfrac{1}{2}[\sin(5x + 3x) + \sin(5x - 3x)] \\ &= \tfrac{1}{2}\sin 8x + \tfrac{1}{2}\sin 2x\end{aligned}$$

11. (a) State the Sum-to-Product Formula for the sum $\sin x + \sin y$.

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

(b) Express $\sin 5x + \sin 7x$ as a product of trigonometric functions.

By the formula in part (a) we have

$$\begin{aligned}\sin 5x + \sin 7x &= 2 \sin \frac{5x + 7x}{2} \cos \frac{5x - 7x}{2} \\ &= 2 \sin 6x \cos(-x) \\ &= 2 \sin 6x \cos x\end{aligned}$$

12. What is a trigonometric equation? How do we solve a trigonometric equation?

A trigonometric equation is an equation involving trigonometric functions. To solve a trigonometric equation, we first find all solutions for one period of the function involved and then add integer multiples of the period to obtain all solutions.

(a) Solve the equation $\cos x = \frac{1}{2}$.

The solutions of this equation in the interval $[0, 2\pi)$ are

$$x = \frac{\pi}{3} \quad \text{and} \quad x = \frac{5\pi}{3}$$

To obtain all solutions, we add multiples of 2π (because $\cos x$ is periodic with period 2π). The solutions are

$$x = \frac{\pi}{3} + 2k\pi \quad \text{and} \quad x = \frac{5\pi}{3} + 2k\pi$$

where k is any integer.

(b) Solve the equation $2 \sin x \cos x = \frac{1}{2}$.

First we use a double-angle formula to express the left-hand side as a single trigonometric function.

$$\begin{aligned}2 \sin x \cos x &= \tfrac{1}{2} \quad \text{Given equation} \\ \sin 2x &= \tfrac{1}{2} \quad \text{Double-Angle Formula}\end{aligned}$$

The solutions of this equation in the interval $[0, 2\pi)$ are

$$2x = \frac{\pi}{6} \quad \text{and} \quad 2x = \frac{5\pi}{6}$$

To obtain all solutions, we add multiples of 2π. The solutions are

$$2x = \frac{\pi}{6} + 2k\pi \quad \text{and} \quad 2x = \frac{5\pi}{6} + 2k\pi$$

and dividing by 2, we get the solutions

$$x = \frac{\pi}{12} + k\pi \quad \text{and} \quad x = \frac{5\pi}{12} + k\pi$$

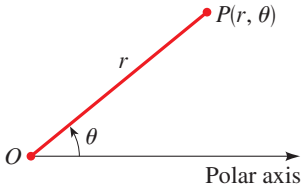
where k is any integer.

Cut here and keep for reference

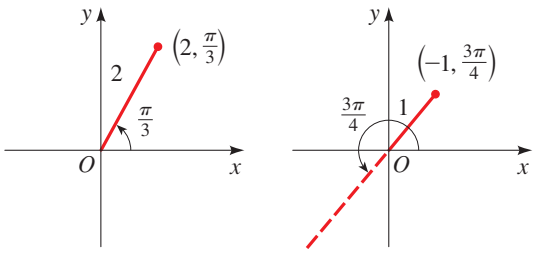
CHAPTER 8 Review: Concept Check Answers

1. (a) Explain the polar coordinate system.

In the **polar coordinate** system the location of a point P in the plane is determined by an ordered pair (r, θ), where r is the distance from the pole O to P and θ is the angle formed by the polar axis and the ray $\overrightarrow{OP}$, as shown in the figure.



(b) Graph the points with polar coordinates $(2, \pi/3)$ and $(-1, 3\pi/4)$.



(c) State the equations that relate the rectangular coordinates of a point to its polar coordinates.

To change from polar to rectangular:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

To change from rectangular to polar:

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

(d) Find rectangular coordinates for $(2, \pi/3)$.

$$x = 2\cos\frac{\pi}{3} = 1 \qquad \text{and} \qquad y = 2\sin\frac{\pi}{3} = \sqrt{3}$$

So in rectangular coordinates the point is $(1, \sqrt{3})$.

(e) Find polar coordinates for $P(-2, 2)$.

$r^2 = (-2)^2 + 2^2 = 8$, so $r = \sqrt{8} = 2\sqrt{2}$.
$\tan\theta = 2/(-2) = -1$ and P is in Quadrant II, so $\theta = 3\pi/4$. So in polar coordinates the point is $(2\sqrt{2}, 3\pi/4)$.

2. (a) What is a polar equation?

A polar equation is an equation in the variables r and θ, where these variables are the polar coordinates of the point (r, θ).

(b) Convert the polar equation $r = \sin\theta$ to an equivalent rectangular equation.

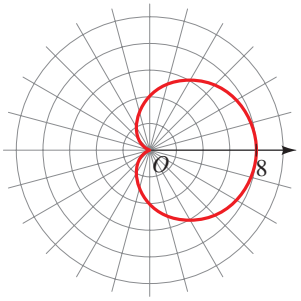
$$r = \sin\theta \qquad \text{Polar equation}$$
$$r^2 = r\sin\theta \qquad \text{Multiply by } r$$
$$x^2 + y^2 = y \qquad \text{Convert}$$

In the last step we substituted $r^2 = x^2 + y^2$ and $r\sin\theta = y$. So an equivalent rectangular equation is $x^2 + y^2 = y$.

3. (a) How do we graph a polar equation?

We plot all the points with polar coordinates (r, θ) that satisfy the equation.

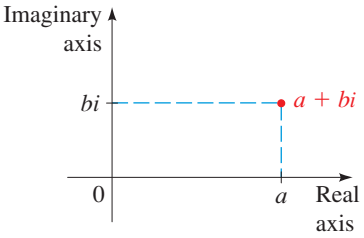
(b) Sketch a graph of the polar equation $r = 4 + 4\cos\theta$. What is the graph called?



This graph is called a cardioid.

4. (a) What is the complex plane? How do we graph a complex number $z = a + bi$ in the complex plane?

The complex plane is a plane determined by two axes: the real axis and the imaginary axis. To graph the complex number $z = a + bi$, we plot the ordered pair (a, b) in this plane as shown.

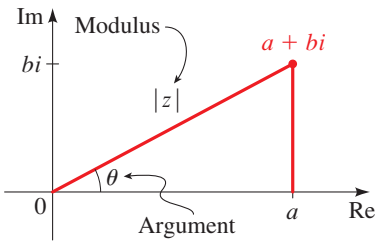


(b) What are the modulus and argument of the complex number $z = a + bi$?

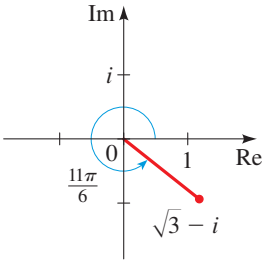
The modulus of z, written $|z|$, is the distance of the point z to the origin in the complex plane. So

$$|z| = \sqrt{a^2 + b^2}$$

The argument of z is the angle θ formed by the line segment connecting the origin to the point z and the positive real axis. So $\tan\theta = b/a$.



(c) Graph the point $z = \sqrt{3} - i$, and find the modulus and argument of z.



Modulus: $|z| = \sqrt{(\sqrt{3})^2 + 1^2}$
$= \sqrt{4} = 2$

Argument: $\theta = \dfrac{11\pi}{6}$

5. (a) How do we express the complex number z in polar form?

The polar form is $z = r(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument of z.

(b) Express $z = \sqrt{3} - i$ in polar form.

Using the moduli and arguments from Question 4(c) above, we get

$$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

6. Let $z_1 = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

and $z_2 = 5\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

(a) Find the product z_1z_2.

To find the product z_1z_2, we multiply the moduli and add the arguments, so

$$z_1z_2 = 10\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right]$$
$$= 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

(b) Find the quotient z_1/z_2.

To find the quotient z_1/z_2, we divide the moduli and subtract the arguments, so

$$\frac{z_1}{z_2} = \frac{2}{5}\left[\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right]$$
$$= \frac{2}{5}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

7. (a) State De Moivre's Theorem.

If $z = r(\cos\theta + i\sin\theta)$ then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

(b) Use De Moivre's Theorem to find the fifth power of $z = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

$$z^5 = 32\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

8. (a) State the formula for the nth roots of a complex number $z = r(\cos\theta + i\sin\theta)$.

The n nth roots are

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

(b) How do we find the nth roots of a complex number?

We use the following guidelines.

1. The modulus of each nth root is $r^{1/n}$.
2. The argument of the first root is θ/n.
3. Add $2\pi/n$ to get the argument of each successive root.

(c) Find the three third roots of $z = -8$.

First we express z in polar form:

$$z = 8(\cos\pi + i\sin\pi)$$

So the modulus of each root is $8^{1/3} = 2$. The argument of the first root is $\pi/3$. We add $2\pi/3$ to get the argument of each successive root. So the three roots are

$$w_0 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = 1 + \sqrt{3}i$$

$$w_1 = 2\left[\cos\left(\frac{\pi + 2\pi}{3}\right) + i\sin\left(\frac{\pi + 2\pi}{3}\right)\right] = -2$$

$$w_2 = 2\left[\cos\left(\frac{\pi + 4\pi}{3}\right) + i\sin\left(\frac{\pi + 4\pi}{3}\right)\right] = 1 - \sqrt{3}i$$

9. (a) What are parametric equations?

Parametric equations are equations of the form

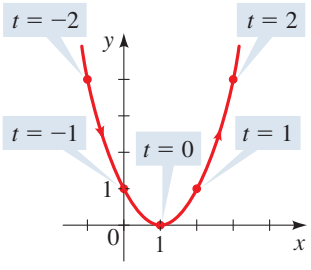
$$x = f(t) \qquad y = g(t)$$

where f and g are functions of the parameter t.

(b) Sketch a graph of the following parametric equations, using arrows to indicate the direction of the curve.

$$x = t + 1 \qquad y = t^2 \qquad -2 \le t \le 2$$

t	(x, y)
−2	(−1, 4)
−1	(0, 1)
0	(0, 0)
1	(2, 1)
2	(3, 4)



(c) Eliminate the parameter to obtain an equation in x and y.

From the first equation $t = x - 1$, so from the second equation we have $y = (x - 1)^2$.

(*continued*)

10. (a) What is a vector in the plane? How do we represent a vector in the coordinate plane?

A vector is a quantity that has both length (or magnitude) and direction. A vector $\mathbf{v}$ in the coordinate plane is expressed in terms of components as

$$\mathbf{v} = \langle a_1, a_2 \rangle$$

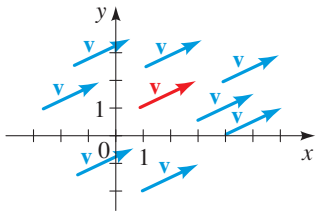
where a_1 is the horizontal component and a_2 is the vertical component.

(b) Find the vector with initial point $(2, 3)$ and terminal point $(4, 10)$.

$$\mathbf{v} = \langle 4 - 2, 10 - 3 \rangle = \langle 2, 7 \rangle$$

(c) Let $\mathbf{v} = \langle 2, 1 \rangle$. If the initial point of $\mathbf{v}$ is placed at $P(1, 1)$, where is its terminal point? Sketch several representations of $\mathbf{v}$.

The terminal point is $Q(1 + 2, 1 + 1) = Q(3, 2)$.



(d) How is the magnitude of $\mathbf{v} = \langle a_1, a_2 \rangle$ defined? Find the magnitude of $\mathbf{w} = \langle 3, 4 \rangle$.

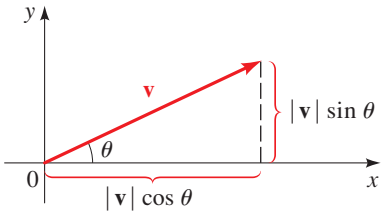
The magnitude of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{a_1^2 + a_2^2}$. We have $|\mathbf{w}| = \sqrt{3^2 + 4^2} = 5$.

(e) What are the vectors $\mathbf{i}$ and $\mathbf{j}$? Express the vector $\mathbf{v} = \langle 5, 9 \rangle$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

The vector $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. So $\mathbf{v} = 5\mathbf{i} + 9\mathbf{j}$.

(f) Let $\mathbf{v} = \langle a_1, a_2 \rangle$ be a vector in the coordinate plane. What is meant by the direction θ of $\mathbf{v}$? What are the coordinates of $\mathbf{v}$ in terms of its length and direction? Sketch a figure to illustrate your answer.

The **direction** of $\mathbf{v}$ is the smallest positive angle θ in standard position formed by the positive x-axis and $\mathbf{v}$. So $\mathbf{v} = |\mathbf{v}| \cos\theta\,\mathbf{i} + |\mathbf{v}| \sin\theta\,\mathbf{j}$.



(g) Suppose that $\mathbf{v}$ has length $|\mathbf{v}| = 5$ and direction $\theta = \pi/6$. Express $\mathbf{v}$ in terms of its coordinates.

$$\mathbf{v} = 5\cos\frac{\pi}{6}\mathbf{i} + 5\sin\frac{\pi}{6}\mathbf{i} = \frac{5\sqrt{3}}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$$

11. (a) Define addition and scalar multiplication for vectors.

Let $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$ and let $c \in \mathbb{R}$. Then $\mathbf{u} + \mathbf{v} = \langle a_1 + b_1, a_2 + b_2 \rangle$, and $c\mathbf{u} = \langle ca_1, ca_2 \rangle$.

(b) If $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle 5, 9 \rangle$, find $\mathbf{u} + \mathbf{v}$ and $4\mathbf{u}$.

$\mathbf{u} + \mathbf{v} = \langle 2 + 5, 3 + 9 \rangle = \langle 7, 12 \rangle$, and $4\mathbf{u} = \langle 4 \cdot 2, 4 \cdot 3 \rangle = \langle 8, 12 \rangle$.

12. (a) Define the dot product of the vectors $\mathbf{u} = \langle a_1, a_2 \rangle$ and $\mathbf{v} = \langle b_1, b_2 \rangle$, and state the formula for the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

The dot product is $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2$. The angle θ satisfies $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$.

(b) If $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle 1, 4 \rangle$, find $\mathbf{u} \cdot \mathbf{v}$ and find the angle between $\mathbf{u}$ and $\mathbf{v}$.

The dot product is $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 3 \cdot 4 = 14$. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies

$$\cos\theta = \frac{14}{\sqrt{13}\sqrt{17}} \approx 0.942$$

So $\theta = \cos^{-1}(0.942) \approx 19.7°$.

(*continued*)

Cut here and keep for reference

CHAPTER 9 Review: Concept Check Answers

1. (a) What are the three methods we use to solve a system of equations?

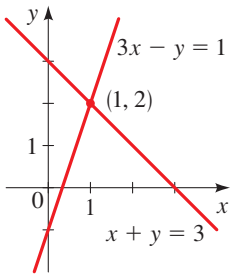
The substitution method, the elimination method, and the graphical method

(b) Solve the system by the elimination method and by the graphical method.

$$\begin{cases} x + y = 3 \\ 3x - y = 1 \end{cases}$$

Elimination method: To eliminate y, we add the two equations to get $4x = 4$, so $x = 1$. Substituting 1 for x in the first equation, we get $1 + y = 3$, so $y = 2$. The solution is $(1, 2)$

Graphical method: We graph the two equations as shown. The point of intersection is $(1, 2)$. So the solution of the system is $(1, 2)$.



2. For a system of two linear equations in two variables:

(a) How many solutions are possible?

Such a system can have one solution, no solution, or infinitely many solutions.

(b) What is meant by an inconsistent system? a dependent system?

A system is inconsistent if it has no solution. A system is dependent if it has infinitely many solutions.

3. What operations can be performed on a linear system so as to arrive at an equivalent system?

1. Add a nonzero multiple of one equation to another.
2. Multiply an equation by a nonzero constant.
3. Interchange the position of two equations.

4. (a) Explain how Gaussian elimination works.

We use the operations in Question 3 above to obtain a system in triangular form and then use back-substitution to solve for the variables.

(b) Use Gaussian elimination to put the following system in triangular form, and then solve the system.

System	Triangular form
$\begin{cases} x + y - 2z = 3 \\ x + 2y + z = 5 \\ 3x - y + 5z = 1 \end{cases}$	$\begin{cases} x + y - 2z = 3 \\ y + 3z = 2 \\ 23z = 0 \end{cases}$

Using back-substitution, we get the solution $(1, 2, 0)$.

5. What does it mean to say that A is a matrix with dimension $m \times n$?

An $m \times n$ matrix A has m rows and n columns.

6. What is the row-echelon form of a matrix? What is a leading entry?

The first nonzero entry in a row (reading from left to right) is called a *leading entry*. A matrix is in row-echelon form if it satisfies the following:

1. The leading entry in each row is 1.
2. The leading entry in each row is to the right of the leading entry in the row above it.
3. All rows consisting entirely of 0's are at the bottom of the matrix.

7. (a) What is the augmented matrix of a system? What are leading variables?

The augmented matrix of a linear system is the matrix that contains the coefficients and the constant terms. A *leading* variable is one that corresponds to a leading entry in the augmented matrix.

(b) What are the elementary row operations on an augmented matrix?

The elementary row operations on a matrix correspond to the operations in Question 3.

1. Add a nonzero multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange the position of two rows.

(c) How do we solve a system using the augmented matrix?

We perform elementary row operations to put the matrix in row-echelon form (as in Question 6). The equations that correspond to the row-echelon form can be solved using back-substitution.

(d) Write the augmented matrix of the following system of linear equations.

$$\begin{cases} x + y - 2z = 3 \\ x + 2y + z = 5 \\ 3x - y + 5z = 1 \end{cases} \qquad \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & 2 & 1 & 5 \\ 3 & -1 & 5 & 1 \end{bmatrix}$$

(e) Solve the system in part (d).

We use elementary row operations to put the augmented matrix into row-echelon form.

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 23 & 0 \end{bmatrix} \qquad \begin{cases} x + y - 2z = 3 \\ y + 3z = 2 \\ 23z = 0 \end{cases}$$

The system is solved by back-substitution as in Question 4(b). The solution is $(1, 2, 0)$.

8. Suppose you have used Gaussian elimination to transform the augmented matrix of a linear system into row-echelon form. How can you tell whether the system has exactly one solution? no solution? infinitely many solutions?

No solution: The row-echelon form has a row that represents the equation $0 = c$, where c is not zero.

Exactly one solution: Each variable in the row-echelon form is a leading variable.

Infinitely many solutions: The variables in the row-echelon form are not all leading.

(*continued*)

CHAPTER 9 Review: Concept Check Answers (*continued*)

9. What is the reduced row echelon form of a matrix?

A matrix is in reduced **row-echelon form** if it is in row-echelon form and also satisfies the following:

Every number above and below each leading entry is a 0.

10. (a) How do Gaussian elimination and Gauss-Jordan elimination differ?

In each method we start with the augmented matrix of a linear system and perform row operations. In Gaussian elimination we put the matrix in row-echelon form. In Gauss-Jordan elimination we put the matrix in reduced row-echelon form.

(b) Use Gauss-Jordan elimination to solve the linear system in part 7(d).

We start with the matrix in 7(d) and continue to use row operations to obtain the following reduced-row echelon form and the corresponding system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{cases} x & & & = 1 \\ & y & & = 2 \\ & & z & = 0 \end{cases}$$

The solution is $(1, 2, 0)$.

11. If A and B are matrices with the same dimension and k is a real number, how do you find $A + B$ and kA?

To find $A + B$, we add corresponding entries. To find kA, we multiply each entry in A by k.

12. (a) What must be true of the dimensions of A and B for the product AB to be defined?

The number of columns of A must be the same as the number of rows of B.

(b) If A has dimension 2×3 and if B has dimension 3×2, is the product AB defined? If so, what is the dimension of AB?

The product AB is defined and has dimension 2×2.

(c) Find the matrix product.

$$\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 5 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 9 & 4 \\ 12 & 16 & 4 \end{bmatrix}$$

13. (a) What is an identity matrix I_n? If A is an $n \times n$ matrix, what are the products AI_n and I_nA?

The identity matrix I_n is an $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere: $AI_n = A$ and $I_nA = A$.

(b) If A is an $n \times n$ matrix, what is its inverse matrix?

The inverse is a matrix A^{-1} with the property that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

(c) Complete the formula for the inverse of a 2×2matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(d) Find the inverse of A.

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

14. (a) Express the system in 1(b) as a matrix equation $AX = B$.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{B}$$

(b) If a linear system is expressed as a matrix equation $AX = B$, how do we solve the system? Solve the system in part (a).

The solution is given by the matrix $X = A^{-1}B$. We found A^{-1} in 13(d). So

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

15. (a) Is it true that the determinant $\det A$ of a matrix A is defined only if A is a square matrix?

Yes

(b) Find the determinant of the matrix A in part 13(d).

$$\det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = (1)(-1) - (1)(3) = -4$$

(c) Use Cramer's Rule to solve the system in 1(b).

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}} = 1 \qquad y = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}} = 2$$

16. (a) How do we express a rational function r as a partial fraction decomposition?

We express r as a sum of fractions whose denominators consist of linear or irreducible quadratic factors.

(b) Give the form of the partial fraction decomposition.

(i) $$\frac{2x}{(x-5)(x-2)^2} = \frac{A}{x-5} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

(ii) $$\frac{2x}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$

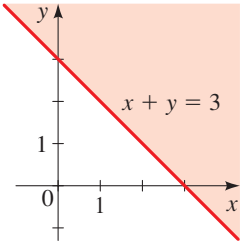
17. (a) How do we graph an inequality in two variables?

We first graph the corresponding equation and then use test points to determine the solution set.

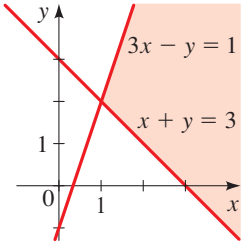
(b) Graph the solution set of the inequality $x + y \geq 3$.

(c) Graph the solution set of the system of inequalities:

$$x + y \geq 3$$
$$3x - y \geq 1$$



$x + y \geq 3$



$x + y \geq 3,\ 3x - y \geq 1$

Cut here and keep for reference

CHAPTER 10 Review: Concept Check Answers

1. (a) Give the geometric definition of a parabola.

A **parabola** is the set of points in the plane that are equidistant from a fixed point F (called the **focus**) and a fixed line l (called the **directrix**).

(b) Give the equation of a parabola with vertex at the origin and with vertical axis. Where is the focus? What is the directrix?

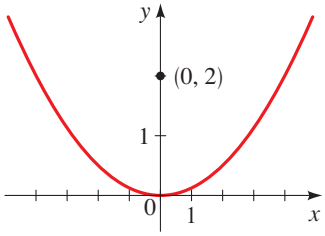
The equation of a parabola with vertical axis and vertex at the origin has the form

$$x^2 = 4py$$

where the focus is $F(0, p)$ and the directrix is the horizontal line $y = -p$.

(c) Graph the equation $x^2 = 8y$. Indicate the focus on the graph.

Writing the equation as $x^2 = 4(2)y$, we see that $p = 2$. So the focus is $F(0, 2)$, and the directrix is the line $y = -2$.



2. (a) Give the geometric definition of an ellipse.

An **ellipse** is the set of all points in the plane, the sum of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are the **foci** (plural of **focus**) of the ellipse.

(b) Give the equation of an ellipse with center at the origin and with major axis along the x-axis. How long is the major axis? How long is the minor axis? Where are the foci? What is the eccentricity of the ellipse?

The equation of an ellipse with center at the origin and with major axis along the x-axis has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

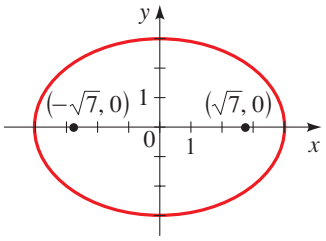
The major axis is along the x-axis, provided that $a > b$. In this case its length is $2a$. The minor axis is along the y-axis, and its length is $2b$.

The foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

The eccentricity is $e = c/a$.

(c) Graph the equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. What are the lengths of the major and minor axes? Where are the foci?

Comparing this equation with the general equation of an ellipse, we see that $a = 4$ and $b = 3$. Since $4 > 3$, the major axis is along the x-axis and has length $2 \cdot 4 = 8$. The minor axis is along the y-axis, and it length is $2 \cdot 3 = 6$. We find $c^2 = 16 - 9 = 7$, so the foci are at $(\pm\sqrt{7}, 0)$.



3. (a) Give the geometric definition of a hyperbola.

A **hyperbola** is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. These two fixed points are the **foci** of the hyperbola.

(b) Give the equation of a hyperbola with center at the origin and with transverse axis along the x-axis. How long is the transverse axis? Where are the vertices? What are the asymptotes? Where are the foci?

The equation of a hyperbola with center at the origin and with transverse axis along the x-axis has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The transverse axis is along the x-axis and its length $2a$.

The vertices are at $(\pm a, 0)$.

The asymptotes are the lines $y = \pm(b/a)x$.

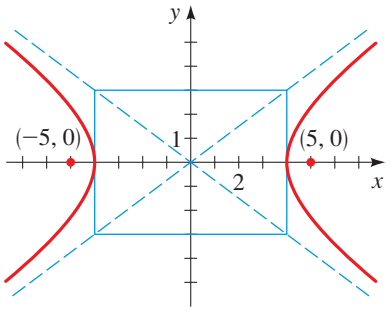
The foci are at $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

(c) What is a good first step in graphing the hyperbola that is described in part (b)?

A good first step is to sketch the central box. This is the rectangle centered at the origin, with sides parallel to the axes, that crosses the x-axis at $\pm a$ and the y-axis at $\pm b$.

(d) Graph the equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. What are the asymptotes? Where are the vertices? Where are the foci? What is the length of the transverse axis?

This is an equation of a hyperbola with transverse axis along the x-axis. The central box crosses the axes at $(\pm 4, 0)$ and $(0, \pm 3)$. The asymptotes are $y = \pm\frac{3}{4}x$. The vertices are $(\pm 4, 0)$. We find $c^2 = 16 + 9 = 25$, so the foci are $(\pm 5, 0)$. Using the vertices and the asymptotes as guides, we graph the hyperbola as shown.



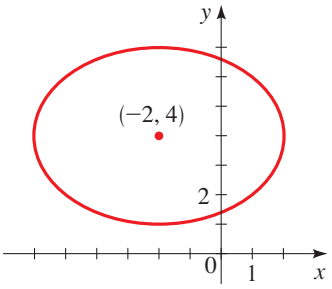
(*continued*)

4. (a) Suppose we are given an equation in x and y. Let h and k be positive numbers. What is the effect on the graph of the equation if x is replaced by $x - h$ or $x + h$ and if y is replaced by $y - k$ or $y + k$?

Replacing x by $x - h$ or $x + h$ shifts the graph to the right or left, respectively, by h units. Replacing y by $y - k$ or $y + k$ shifts the graph upward or downward respectively, by k units.

(b) Sketch a graph of $\dfrac{(x+2)^2}{16} + \dfrac{(y-4)^2}{9} = 1$.

The graph is the same as the ellipse in 2(c) but shifted left 2 units and upward 4 units. So the center of the ellipse is at $(-2, 4)$.



5. (a) How can you tell whether the following nondegenerate conic is a parabola, an ellipse, or a hyperbola?

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

The graph is a parabola if either A or C is 0, an ellipse if A and C have the same sign (a circle if $A = C$), or a hyperbola if A and C have opposite signs.

(b) What conic does $3x^2 - 5y^2 + 4x + 5y - 8 = 0$ represent?

The graph is a hyperbola because the coefficients of x^2 and y^2 (3 and -5) have opposite signs.

6. (a) Suppose that the x- and y-axes are rotated through an acute angle ϕ to produce the X- and Y-axes. What are the equations that relate the coordinates (x, y) and (X, Y) of a point in the xy-plane and XY-plane, respectively?

$$x = X\cos\phi - Y\sin\phi \qquad X = x\cos\phi + y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi \qquad Y = -x\sin\phi + y\cos\phi$$

(b) In the equation below, how do you eliminate the xy-term?

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Rotate the axes through an angle ϕ that satisfies

$$\cot 2\phi = \frac{A - C}{B}$$

(c) Use a rotation of axes to eliminate the xy-term in the equation $25x^2 - 14xy + 25y^2 = 288$. Graph the equation.

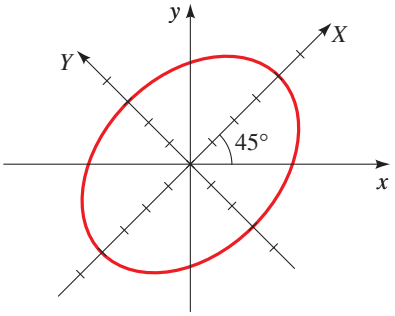
The angle ϕ satisfies $\cot 2\phi = 0$, so $\phi = 45°$. By part (a) we have

$$x = X\cos 45° - Y\sin 45° = \frac{X - Y}{\sqrt{2}}$$

$$y = X\sin 45° + Y\cos 45° = \frac{X + Y}{\sqrt{2}}$$

Substituting into the given equation and simplifying, we get

$$\frac{X^2}{16} + \frac{Y^2}{9} = 1$$



7. (a) What is the discriminant of the equation in 6(b)? How can you use the discriminant to determine the type of conic that the equation represents?

The discriminant is $B^2 - 4AC$. The conic is a parabola, ellipse, or hyperbola provided that the discriminant is zero, negative, or positive, respectively.

(b) Use the discriminant to identify the equation in 6(c).

The discriminant is $14^2 - 4(25)(25) < 0$, which confirms that the equation represents an ellipse.

8. (a) Write polar equations that represent a conic with eccentricity e. For what values of e is the conic an ellipse? a hyperbola? a parabola?

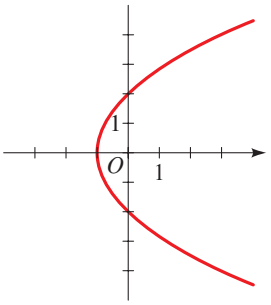
Polar equations of conics have the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

The equation represents a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

(b) What conic does the polar equation $r = 2/(1 - \cos\theta)$ represent? Graph the conic.

This is a polar equation of a parabola.



Cut here and keep for reference

CHAPTER 11 Review: Concept Check Answers

1. (a) What is a sequence? What notation do we use to denote the terms of a sequence?

A sequence is a list of numbers written in a specific order. Each number is called a term of the sequence. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots$.

(b) Find a formula for the sequence of even numbers and a formula for the sequence of odd numbers.

Even numbers: $a_n = 2n$
Odd numbers: $a_n = 2n + 1$

(c) Find the first three terms and the 10th term of the sequence given by $a_n = n/(n + 1)$.

$$a_1 = \tfrac{1}{2}, \quad a_2 = \tfrac{2}{3}, \quad a_3 = \tfrac{3}{4}, \quad \text{and} \quad a_{10} = \tfrac{10}{11}$$

2. (a) What is a recursively defined sequence?

A recursively defined sequence is a sequence in which each term depends on some or all of the preceding terms.

(b) Find the first four terms of the sequence recursively defined by $a_1 = 3$ and $a_n = n + 2a_{n-1}$.

$$a_1 = 3, \quad a_2 = 8, \quad a_3 = 19, \quad a_4 = 42$$

3. (a) What is meant by the partial sums of a sequence?

The nth partial sum S_n of a sequence $a_1, a_2, a_3, \ldots$ is obtained by adding the first n terms of the sequence $S_1 = a_1, S_2 = a_1 + a_2, \ldots$, and in general $S_n = a_1 + a_2 + \cdots + a_n$.

(b) Find the first three partial sums of the sequence given by $a_n = 1/n$.

$$S_1 = \tfrac{1}{1} = 1, \quad S_2 = \tfrac{1}{1} + \tfrac{1}{2} = \tfrac{3}{2}, \quad S_3 = \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{11}{6}$$

4. (a) What is an arithmetic sequence? Write a formula for the nth term of an arithmetic sequence.

An arithmetic sequence a_n is obtained when we start with a number a and add to it a fixed constant d over and over again. So

$$a_n = a + (n - 1)d$$

(b) Write a formula for the arithmetic sequence that starts as follows: 3, 8, . . . Write the first five terms of this sequence.

The first term is $a = 3$, and the common difference is 5. So the nth term is

$$a_n = 3 + (n - 1)5$$

which simplifies to $a_n = -2 + 5n$. So the first five terms are 3, 8, 13, 18, and 23.

(c) Write two different formulas for the sum of the first n terms of an arithmetic sequence.

$$S_n = \frac{n}{2}[2a + (n - 1)d] \quad \text{and} \quad S_n = n\left[\frac{a + a_n}{2}\right]$$

(d) Find the sum of the first 20 terms of the sequence in part (b).

Using the first formula in part (c), we get

$$S_{20} = \frac{20}{2}[2 \cdot 3 + 19 \cdot 5] = 1010$$

5. (a) What is a geometric sequence? Write an expression for the nth term of a geometric sequence that has first term a and common ratio r.

A geometric sequence a_n is obtained when we start with a number a and multiply it by a fixed constant r over and over again. So

$$a_n = ar^{n-1}$$

(b) Write an expression for the geometric sequence with first term $a = 3$ and common ratio $r = \frac{1}{2}$. Give the first five terms of this sequence.

The nth term is $a_n = 3\left(\frac{1}{2}\right)^{n-1}$.

The first five terms are 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, and $\frac{3}{16}$.

(c) Write an expression for the sum of the first n terms of a geometric sequence.

$$S_n = a\frac{1 - r^n}{1 - r}$$

(d) Find the sum of the first five terms of the sequence in part (b).

Using the formula in part (c), we get

$$S_5 = 3\frac{1 - \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)} = \frac{93}{16}$$

6. (a) What is an infinite geometric series?

An infinite geometric series is a series with infinitely many terms of the form

$$a + ar + ar^2 + \cdots + ar^n + \cdots$$

(b) What does it mean for an infinite series to converge? For what values of r does an infinite geometric series converge? If an infinite geometric series converges, then what is its sum?

An infinite series converges if its sequence of partial sums S_n approaches a finite number as $n \to \infty$. An infinite geometric series converges if $|r| < 1$; in this case its sum is $S = a/(1 - r)$.

(c) Write the first four terms of the infinite geometric series with first term $a = 5$ and common ratio $r = 0.4$. Does the series converge? If so, find its sum.

$$5 + 5(0.4) + 5(0.4)^2 + 5(0.4)^3 + \cdots + 5(0.4)^n + \cdots$$

The series converges because $|0.4| < 1$. By the formula in part (b) the sum of the series is

$$S = \frac{5}{1 - 0.4} = \frac{25}{3}$$

7. (a) Write $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ using sigma notation.

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \sum_{k=1}^{5} k^3$$

(b) Write $\sum_{k=3}^{5} 2k^2$ without using sigma notation.

$$\sum_{k=3}^{5} 2k^2 = 2 \cdot 3^2 + 2 \cdot 4^2 + 2 \cdot 5^2$$

(*continued*)

8. (a) What is an annuity? Write an expression for the amount A_f of an annuity consisting of n regular equal payments of size R with interest rate i per time period.

An annuity is a sum of money that is paid in regular equal payments. The amount is

$$A_f = R\frac{(1+i)^n - 1}{i}$$

(b) An investor deposits \$200 each month into an account that pays 6% compounded monthly. How much is in the account at the end of 3 years?

The interest per time period is $i = 0.06/12 = 0.005$; the number of time periods is $n = 3 \times 12 = 36$.

$$A_f = 200\frac{(1+0.005)^{36} - 1}{0.005} = 7867.22$$

(c) What is the formula for calculating the present value of the annuity in part (b)?

$$A_p = R\frac{1-(1+i)^{-n}}{i}$$

(d) What is the present value of the annuity in part (b)?

$$A_p = 200\frac{1-(1+0.005)^{-36}}{0.005} = 6574.20$$

(e) When buying on installment, what is the formula for calculating the periodic payments?

If a loan A_p is to be repaid in n regular equal payments with interest rate i per time period, then the size R of each payment is

$$R = \frac{iA_p}{1-(1+i)^{-n}}$$

(f) If you take out a 5-year loan for \$10,000 at 3% interest compounded monthly, what is the size of each monthly payment?

The interest per time period is $i = 0.03/12 = 0.0025$; the number of time periods is $n = 5 \times 12 = 60$.

$$R = \frac{0.0025(10{,}000)}{1-(1+0.0025)^{-60}} = 179.69$$

9. (a) State the Principle of Mathematical Induction.

A statement $P(n)$ about a natural number n is true for all n provided that the following hold.

1. $P(1)$ is true.
2. If $P(k)$ is true, then $P(k+1)$ is true.

(b) Use mathematical induction to prove that for all natural numbers n, $3^n - 1$ is an even number.

Let $P(n)$ be the statement that $3^n - 1$ is even.

1. $P(1)$ is true because $3^1 - 1$ is even.
2. Suppose $P(k)$ is true. Now

$$3^{k+1} - 1 = 3\cdot 3^k - 1 = \underbrace{2\cdot 3^k}_{\text{even}} + \underbrace{3^k - 1}_{\text{even}}$$

The first term is clearly even, and the second is even by the induction hypothesis. So $P(k+1)$ is true.

It follows that the statement is true for all n.

10. (a) Write Pascal's triangle. How are the entries in the triangle related to each other?

Row 0 1
Row 1 1 1
Row 2 1 2 1
Row 3 1 3 3 1

Each entry is the sum of the two entries above it.

(b) Use Pascal's triangle to expand $(x+c)^3$.

$$(x+c)^3 = x^3 + 3x^2c + 3xc^2 + c^3$$

11. (a) What does the symbol $n!$ mean? Find 5!.

$$n! = 1\cdot 2\cdot 3\cdot\cdots\cdot n; \quad 5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$$

(b) Define $\binom{n}{r}$, and find $\binom{5}{2}$.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \qquad \binom{5}{2} = \frac{5!}{2!\,3!} = 10$$

12. (a) State the Binomial Theorem.

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n$$

(b) Use the Binomial Theorem to expand $(x+2)^3$.

$$(x+2)^3 = x^3 + \binom{3}{1}x^2 2^1 + \binom{3}{2}x^1\cdot 2^2 + 2^3$$
$$= x^3 + 6x^2 + 12x + 8$$

(c) Use the Binomial Theorem to find the term containing x^4 in the expansion of $(x+2)^{10}$.

The term is $\binom{10}{4}x^4 2^6 = 13440x^4$.

Cut here and keep for reference

CHAPTER 12 Review: Concept Check Answers

1. (a) Explain what is meant by $\lim_{x \to a} f(x) = L$.

As x approaches a, the values of $f(x)$ approach the number L.

(b) If $\lim_{x \to 2} f(x) = 5$, is it possible that $f(2) = 3$?

Yes. For x close to 2, $f(x)$ is close to 5. But the value of f at 2 does not affect the limit.

(c) Find $\lim_{x \to 2} x^2$.

If x is close to 2, then x^2 is close to 4, so $\lim_{x \to 2} x^2 = 4$.

2. To evaluate the limit of a function, we often need to first rewrite the function using the rules of algebra. What is the logical first step in evaluating each of the following limits?

(a) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

Factor the numerator, and simplify.

(b) $\lim_{h \to 0} \dfrac{(5 + h)^2 - 25}{h}$

Expand the numerator, and simplify.

(c) $\lim_{x \to 3} \dfrac{\sqrt{x + 1} - 2}{x - 3}$

Rationalize the numerator by multiplying the numerator and denominator by $\sqrt{x + 1} + 2$, and then simplify.

(d) $\lim_{x \to 7} \dfrac{\left(\frac{1}{7} - \frac{1}{x}\right)}{x - 7}$

Combine the fractions in the numerator, using the common denominator $7x$, and simplify.

3. (a) Explain what it means to say:

$$\lim_{x \to 3^-} f(x) = 5 \qquad \lim_{x \to 3^+} f(x) = 10$$

The first equation says that the limit, as x approaches 3 from the left (through values to the left of 3, or less than 3), is 5.

The second equation says that the limit, as x approaches 3 from the right (through values to the right of 3, or greater than 3), is 10.

(b) If the two equations in part (a) are true, is it possible that $\lim_{x \to 3} f(x) = 5$?

No. For the limit to exist, the left- and right-hand limits must be equal.

(c) Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$, where f is defined as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \le 2 \\ x & \text{if } x > 2 \end{cases}$$

As x gets close to 2 from the left (through values less than 2), the function is constantly 1, so the limit from the left is 1. Similarly, as x gets close to 2 from the right (through values greater than 2), the function is equal to x, so the values of f approach 2. Thus the limit from the right is 2.

$$\lim_{x \to 2^-} f(x) = 1 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 2$$

(d) For f as in (c), does $\lim_{x \to 2} f(x)$ exist?

No, because the left- and right-hand limits are not equal.

4. (a) Define the derivative $f'(a)$ of a function f at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

(b) State an equivalent formulation for $f'(a)$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(c) Find the derivative of $f(x) = x^2$ at $x = 3$.

Using the definition in (a), we have

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\ &= \lim_{h \to 0} \frac{(3 + h)^2 - 3^2}{h} \\ &= \lim_{h \to 0} \frac{6h + h^2}{h} \\ &= \lim_{h \to 0} (6 + h) = 6 \end{aligned}$$

So $f'(3) = 6$.

5. (a) Give two different interpretations of the derivative of the function $y = f(x)$ at $x = a$.

The derivative has the following interpretations:

$f'(a)$ is the slope of the tangent line to the graph of f at the point $P(a, f(a))$.

$f'(a)$ is the instantaneous rate of change of y with respect to x at $x = a$.

(b) For the function $f(x) = x^2$, find the slope of the tangent line to the graph of f at the point $(3, 9)$ on the graph.

The slope is $f'(3) = 6$ (from 4(c)).

(c) For the function $y = x^2$, find the instantaneous rate of change of y with respect to x when $x = 3$.

The instantaneous rate of change is $f'(3) = 6$ (from 4(c)).

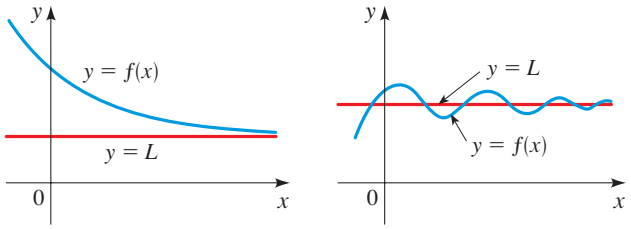
(d) Write expressions for the average rate of change of y with respect to x between a and x and for the instantaneous rate of change of y with respect to x at $x = a$.

Average rate of change	Instantaneous rate of change
$\dfrac{f(x) - f(a)}{x - a}$	$\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

(*continued*)

6. (a) Explain what is meant by $\lim_{x\to\infty} f(x) = L$. Draw sketches to illustrate different ways in which this can happen.

It means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.



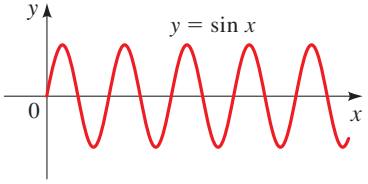
(b) Find $\lim_{x\to\infty} \dfrac{3x^2 + x}{x^2 + 1}$.

We divide numerator and denominator by x^2.

$$\lim_{x\to\infty} \frac{3x^2 + x}{x^2 + 1} = \lim_{x\to\infty} \frac{3 + \dfrac{1}{x}}{1 + \dfrac{1}{x^2}} = \frac{3 + 0}{1 + 0} = 3$$

(c) Explain why $\lim_{x\to\infty} \sin x$ does not exist.

From the graph below we see that as x increases, the values of the sine function oscillate between 1 and -1, so they don't approach a definite number.



7. (a) If $a_1, a_2, a_3, \ldots$ is a sequence, what is meant by $\lim_{n\to\infty} a_n = L$? What is a convergent sequence?

It means that the nth term a_n of the sequence can be made arbitrarily close to L by taking n sufficiently large. The sequence is convergent if $\lim_{n\to\infty} a_n$ exists.

(b) Find $\lim_{n\to\infty} (-1)^n/n$.

As n gets large, the denominator gets large but the numerator is always 1 or -1, so the limit is 0.

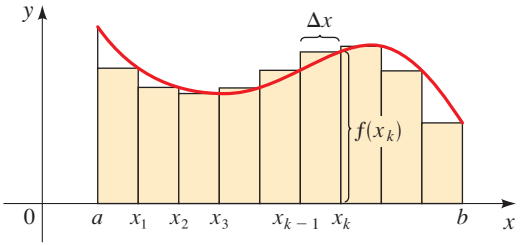
8. (a) Suppose S is the region under the graph of the function $y = f(x)$ and above the x-axis, where $a \le x \le b$. Explain how this area is approximated by rectangles, and write an expression for the area of S as a limit of sums.

We first divide the interval $[a, b]$ into n equal intervals, each of length $\Delta x = (b - a)/n$, and then erect a rectangle on each of these intervals, where its height is the value of f at the right-hand endpoint. The area is the limit of the sum of these approximating rectangles:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

where

$$\Delta x = \frac{b - a}{n} \qquad \text{and} \qquad x_k = a + k\,\Delta x$$



(b) Find the area under the graph of $f(x) = x^2$ and above the x-axis, between $x = 0$ and $x = 3$.

We first divide the interval $[0, 3]$ into n equal intervals, each of length

$$\Delta x = \frac{3 - 0}{n} = \frac{3}{n}$$

The right-hand endpoint of the kth interval is

$$x_k = 0 + k\frac{3}{n} = \frac{3k}{n}$$

So the area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^2\cdot\frac{3}{n} = \lim_{n\to\infty} \frac{3^3}{n^3}\sum_{k=1}^{n} k^2 \\ &= \lim_{n\to\infty} \frac{27}{n^3}\cdot\frac{n(n + 1)(2n + 1)}{6} \\ &= \lim_{n\to\infty} \frac{54n^3 + 81n^2 + 27n}{6n^3} = 9 \end{aligned}$$